ON OSCILLATIONS OF BIHARMONIC FUNCTIONS IN A DISK

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Abstract

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ON OSCILLATIONS OF BIHARMONIC FUNCTIONS IN A DISK

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The basic biharmonic problem for a given plane domain G, as is known (1), consists in determining a function $u(x, y)$ having in the domain G continuous partial derivatives up to the fourth order inclusive, which inside G satisfy the equation

$$\partial^4 u/\partial x^4 + 2\partial^4 u/\partial x^2 \partial y^2 + \partial^4 u/\partial y^4 = 0,$$

and on the boundary of the domain G the conditions

$$u\big|_\Gamma = f(s), \qquad \partial u/\partial n\big|_\Gamma = h(s),$$

where n is the direction of the normal outward to Γ; $f(s)$ and $h(s)$ are given functions of the arc s of the contour Γ. Here the case is considered when G is a disk, $r = \sqrt{x^2 + y^2} < 1$, $x = r\cos\varphi$, $y = r\sin\varphi$, and $h(\varphi) \equiv 0$.

Denote by $u_f(r, \varphi)$ the solution of the biharmonic problem corresponding to the function $f(\varphi)$ prescribed on the boundary of the disk G, under the condition that $\partial u_f(r, \varphi)/\partial r\big|_{r=1} = 0$.

In the present note the question is investigated of the magnitude of the oscillations of the biharmonic function $u_f(r, \varphi)$ at arbitrary two points (r_1, φ) and (r_2, φ) inside the disk (along radii), and also for arbitrary r along arcs of circles concentric with G.

Theorem 1. *Whatever the continuous, 2π-periodic function $f(\varphi)$, for $0 \le r_1, r_2 \le 1$ and for any $0 \le \varphi < 2\pi$ the inequality holds*

$$|u_f(r_2, \varphi) - u_f(r_1, \varphi)| \le \begin{cases} c_1 \displaystyle\int_{r_1}^{r_2} \frac{\omega_2(1-r)}{1-r}\, dr, & \text{if } r_2 - r_1 < 1 - r_2, \\ c_2 \omega_2(r_2 - r_1), & \text{if } r_2 - r_1 \ge 1 - r_2, \end{cases} \tag{1}$$

where c_1, c_2 are certain absolute constants,

$$\omega_2(t) = \sup_{0\le h\le t} |f(\varphi+h) - 2f(\varphi) + f(\varphi-h)|$$

is the modulus of smoothness of the function $f(\varphi)$.

From Theorem 1, for $r_2 = 1$, follows the inequality $|f(\varphi) - f(r,\varphi)| \le c\omega_2(1-r)$, obtained earlier by S. Kaniev (see (^{2}), Theorem 1).

Consider the class $W^{(p)}$ ($p \ge 1$ an integer) of 2π-periodic functions $f(\varphi)$ having an absolutely continuous derivative of order $p-1$ and such that $|f^{(p)}(\varphi)| \le 1$ almost everywhere.

For biharmonic functions inside the disk $u_f(r,\varphi)$, corresponding to functions $f(\varphi) \in W^{(p)}$ under the condition that $\partial u_f(r,\varphi)/\partial r\big|_{r=1} = 0$, the following holds.

Theorem 2. *Whatever the function $f(\varphi) \in W^{(p)}$, for any values r_1, r_2 ($0 \le r_1, r_2 \le 1$) the exact equality holds*

$$\sup_{f\in W^{(p)}} \max_{0\le\varphi\le 2\pi} |u_f(r_2,\varphi) - u_f(r_1,\varphi)| = \frac{4}{\pi}\left|\sum_{\nu=0}^{\infty}(-1)^{\nu(p+1)}\frac{r_2^{2\nu+1} - r_1^{2\nu+1}}{(2\nu+1)^{p+1}} + \right. \qquad (2)$$

$$\left. + \frac{1-r_2^2}{2}\sum_{\nu=0}^{\infty}(-1)^{\nu(p+1)}\frac{r_2^{2\nu+1}}{(2\nu+1)^p} - \frac{1-r_1^2}{2}\sum_{\nu=0}^{\infty}(-1)^{\nu(p+1)}\frac{r_1^{2\nu+1}}{(2\nu+1)^p}\right|.$$

In the particular case when $r_2 = 1$, the assertion of Theorem 2 was obtained in (2).

For an arbitrary natural value of k, denote by

$$\Delta_h^k u_f(r,\varphi) = \sum_{\nu=0}^{k}(-1)^{\nu}\binom{k}{\nu}u_f(r,\varphi+\nu h) \qquad (3)$$

the difference of the function $u_f(r,\varphi)$ with respect to φ of order k with step h.

Theorem 3. Whatever the function $f(\varphi) \in W^{(p)}$ and the natural $k \le p+1$, for every positive $h \le \pi$, everywhere in the disk the sharp inequality

$$\left|\Delta_h^k u_f(r,\varphi)\right| \le \frac{4}{\pi}\left|\sum_{\nu=0}^{\infty}(-1)^{\nu(p+k+1)}r^{2\nu+1}\frac{\{2\sin((2\nu+1)h/2)\}^k}{(2\nu+1)^{p+1}} + \right. \qquad (4)$$

$$\left. + \frac{1-r^2}{2}\sum_{\nu=0}^{\infty}(-1)^{\nu(p+k+1)}r^{2\nu+1}\frac{\{2\sin((2\nu+1)h/2)\}^k}{(2\nu+1)^p}\right|.$$

Inequality (4) holds for all $0 \leq r < 1,\ 0 \leq \varphi < 2\pi$, and also when $p = 0$, i.e., for every function $f(\varphi)$ satisfying the condition $|f(\varphi)| \leq 1$, if $k = 1$.

Considering the class $W^{(p)}L$ ($p \geq 1$ an integer) of 2π-periodic functions having an absolutely continuous derivative of order $p-1$ and such that

$$\int_0^{2\pi} |f^{(p)}(\varphi)|\, d\varphi \leq 1,$$

one can also obtain sharp inequalities for the mean oscillations along radii and along circles concentric with G of the solutions of the corresponding biharmonic problem.

Theorem 4. For all natural p and arbitrary r_1, r_2 $(0 \leq r_1, r_2 \leq 1)$, the equality

$$\sup_{f\in W^{(p)}L} \int_0^{2\pi} |u_f(r_2,\varphi) - u_f(r_1,\varphi)|\, d\varphi = \frac{4}{\pi}\left|\sum_{\nu=0}^{\infty}(-1)^{\nu(p+1)}\frac{r_2^{2\nu+1} - r_1^{2\nu+1}}{(2\nu+1)^{p+1}} + \right. \tag{5}$$

$$\left. + \frac{1-r_2^2}{2}\sum_{\nu=0}^{\infty}(-1)^{\nu(p+1)}\frac{r_2^{2\nu+1}}{(2\nu+1)^p} - \frac{1-r_1^2}{2}\sum_{\nu=0}^{\infty}(-1)^{\nu(p+1)}\frac{r_1^{2\nu+1}}{(2\nu+1)^p}\right|$$

holds.

The particular case of equality (4), when $r_2 = 1$, was obtained in (3).

Theorem 5. For all natural k and p $(k \leq p+1)$, for arbitrary values of r and h $(0 \leq r < 1,\ 0 < h \leq \pi)$, the equality

$$\sup_{f\in W^{(p)}L} \int_0^{2\pi} |\Delta_h^k u_f(r,\varphi)|\, d\varphi = \frac{4}{\pi}\left|\sum_{\nu=0}^{\infty}(-1)^{\nu(p+k+1)} r^{2\nu+1}\frac{\{2\sin((2\nu+1)h/2)\}^k}{(2\nu+1)^{p+1}} + \right. \tag{6}$$

$$\left. + \frac{1-r^2}{2}\sum_{\nu=0}^{\infty}(-1)^{\nu(p+k+1)} r^{2\nu+1}\frac{\{2\sin((2\nu+1)h/2)\}^k}{(2\nu+1)^p}\right|$$

holds.

Equality (6) for $0 \leq r < 1$, when $k = 1,\ p = 0$, is also true in the case where the class $W^{(p)}L$ is replaced by the class of all functions satisfying the conditions

$$\int_0^{2\pi} f(t)\, dt = 0 \quad \text{and} \quad \int_0^{2\pi} |f(t)|\, dt \leq 1.$$

We note that problems analogous to those considered here (Theorems 2–5) for functions harmonic in a disk were studied in the work [4]. In the study of oscillations of the solution of the biharmonic problem, additional difficulties arise, connected with the more complicated character of the behavior of the kernel in the corresponding integral representation for differences (1) and (3).

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REFERENCES

1. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity*, Publishing House of the Academy of Sciences of the USSR, 1954.
2. S. Kanev, DAN, 153, No. 5 (1963).
3. S. Kanev, Reports of the Academy of Sciences of the Ukrainian SSR, No. 4 (1964).
4. A. F. Timan, V. N. Trofimov, DAN, 173, No. 4 (1967).

Note: Figure translations are in progress. See original paper for figures.