Reports of the Academy of Sciences of the USSR

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR
1968. Vol. 182, No. 6

UDC 517.946.6

MATHEMATICS

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ON DIRICHLET SERIES FOR ELLIPTIC OPERATORS

(Presented by Academician A. N. Tikhonov on 29 II 1968)

In the present article we study properties of the analytic continuation of the Dirichlet series composed of the eigenvalues of an elliptic operator. This problem for the Laplace operator was considered in papers ([1–5]).

Let g be an arbitrary bounded N-dimensional domain with sufficiently smooth boundary Γ.

1°. Consider in the domain $(g+\Gamma)$ the following eigenfunction problem:

$$\Delta u + \lambda u = 0 \quad \text{in the domain } g,$$

$$u|_\Gamma = 0 \quad \text{or} \quad \partial u/\partial n|_\Gamma = 0. \tag{1}$$

Here Δ is the Laplace operator, $\{\lambda_i\}$ are the eigenvalues, and $\{u_i\}$ are the orthonormal eigenfunctions of problem (1).

Theorem 1. The Dirichlet series

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i^z},$$

convergent for $\operatorname{Re} z > N/2$, admits an analytic continuation $\varphi(z)$ to the entire complex plane; moreover, the formula

$$\varphi(z) = \sum_{k=1}^{N} \frac{a_k}{z-k/2} + \sum_{k=2}^{[(s+1)/2]} \frac{b_k}{z+(2k-3)/2} + \psi(z); \tag{2}$$

holds, where s is an arbitrary natural number, and the function $\psi(z)$ is regular for

$$\operatorname{Re} z > -(s-1)/2.$$

Thus, formula (2) makes it possible to indicate explicitly all singularities of the function $\varphi(z)$ in any finite domain of the complex plane.

Theorem 1, both in result and in method, is a generalization of Pleijel' s papers ([4,5]), in which he considered the cases of 2 and 3 dimensions ($N = 2, 3$). The proof of Theorem 1 is based on the following two facts (see below formulas (5) and (6)). Denote by $G(x, y, \varkappa^2)$ the Green' s function of the problem

$$\Delta u - \varkappa^2 u = f \quad \text{in the domain } g,$$

$$u|_\Gamma = 0 \quad \text{or} \quad \partial u/\partial n|_\Gamma = 0. \tag{3}$$

The Green' s function $G(x, y, \varkappa^2)$ is representable in the form

$$G(x, y, \varkappa^2) = \frac{\varkappa^{(N-2)/2}}{a_N r_{xy}^{(N-2)/2}} K_{(N-2)/2}(\varkappa r_{xy}) - \gamma(x, y, \varkappa), \tag{4}$$

where $a_N > 0$ is a constant; $K_{(N-2)/2}(\varkappa r_{xy})$ is a Macdonald function; $\gamma(x, y, \varkappa)$ is a function ensuring fulfillment of the boundary condition in (3). For

an asymptotic formula has been established for $\gamma(x, y, \chi)$

$$\gamma(x, x, \chi) = \sum_{k,l,p} C_{klp}(\xi)\, \eta^p \chi^{(N-2)/2+k} \left[\frac{d}{d\eta}\right]^l \left\{\frac{K_{(N-2)/2+k}(2\chi\eta)}{\eta^{(N-2)/2+k}}\right\} + O(\chi^{-s} e^{-2a\chi\eta}). \tag{5}$$

Here the sum contains a finite number of terms; the degree in η of each of them is not less than $-(N-2)$; η is the distance from the point x to the boundary Γ; ξ is a point on Γ; s is any natural number; $a < 1$.

In addition, a formula has been established which expresses the difference of the Green function of problem (3) for two different values of χ through a bilinear series in the eigenfunctions of problem (1). (This formula generalizes Carleman' s formula, valid for $N \leqslant 3$, see ([1]).)

$$G(x, y, \chi^2) - \sum_{k=0}^{n-1} C_{n-1}^k (\chi^2 - \chi_0^2)^k G^{(k+1)}(x, y, \chi_0^2) =$$

$$= (\chi^2 - \chi_0^2)^n \sum_{i=1}^{\infty} \frac{(\lambda_i + 2\chi_0^2 - \chi^2)^{n-1} u_i(x) u_i(y)}{(\lambda_i + \chi_0^2)^n \left[(-1)^n (\lambda_i + \chi_0^2)^n - (\chi^2 - \chi_0^2)^n\right]}; \tag{6}$$

here $n = [N/2]$; $G^{(k)}$ is the k-th iteration of the Green function $G(x, y, \chi_0^2)$.

2°. Consider in the domain $(g + \Gamma)$ the eigenfunction problem for an elliptic system of differential equations of order $2m$ with real coefficients

$$Au \equiv \sum_{|k| \leqslant 2m} A_k(x) D^k u = -\lambda u \quad \text{in the domain } g,$$

$$B_\nu(x, D)u\big|_\Gamma = \sum_{j=1}^{r} \sum_{|k| \leqslant r_\nu} b_{\nu j}^{(k)}(x) D^k u^j\big|_\Gamma = 0; \tag{7}$$

$$\nu = 1, 2, \dots, mr; \qquad r_\nu \leqslant 2m - 1; \qquad u = (u^1, u^2, \dots, u^r) \text{ is an } r\text{-vector}.$$

We shall assume that the operator A, with domain of definition

$$D(A) = \{u;\ u \in C^{(2m)}(g + \Gamma),\ B_\nu(x, D)u = 0\},$$

is formally self-adjoint and bounded below, i.e.

$$(Au, u) \geqslant a\|u\|^2_{W_2^m};$$

the system $\{A, B_\nu\}$ satisfies the Lopatinskii condition, and the coefficients of the system $A_k(x)$ and $b_{\nu j}^{(k)}(x)$ are sufficiently smooth (more precisely, they must belong to the classes indicated in Theorem 1 of ([6])).

Theorem 2. Let $\{\lambda_i\}$ and $\{u_i\}$ be the eigenvalues and the orthonormal eigenfunctions of the unique self-adjoint extension in L_2 of the operator A. The Dirichlet series

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i^z},$$

which converges for $\operatorname{Re} z > N/2m$, has an analytic continuation $\varphi(z)$, equal to

$$\varphi(z) = \frac{C}{z - N/2m} + \psi(z), \tag{8}$$

where the function $\psi(z)$ is regular for $\operatorname{Re} z > (N-1)/2m$.

From (8), with the aid of a Tauberian theorem, one obtains the asymptotic formula for the number $\theta(\lambda)$ of eigenvalues of problem (7) not exceeding λ^*:

$$\theta(\lambda) \sim c_0 \lambda^{N/2m}. \tag{9}$$

This last result, even in a more general situation, was obtained by Browder [7] by another method.

* We note that the constants in formulas (2), (8), and (9) can be written explicitly.

The proof of Theorem 2 is carried out by the method of Minakshisundaram [(2)], based in our case on representing the Green matrix of the parabolic problem corresponding to problem (7) in the form of a bilinear series in the eigenfunctions of problem (7)

$$G(x, y, t) = \sum_{i=1}^{\infty} e^{-\lambda_i t}\, u_i(x) \otimes u_i(y).$$

The estimates of the Green matrix in the closed cylinder $(g+\Gamma)\times[0, T]$ needed for carrying out this method were established in the work of Ivasišen and Eidelman [(6)].

Let us note that, along the way, we obtain the following result: the bilinear series of the form

$$\sum_{i=1}^{\infty} \frac{u_i(x) \otimes u_i(y)}{\lambda_i^{N/2m+\varepsilon}},$$

where $\varepsilon > 0$ is arbitrary, converges absolutely and uniformly in the closed domain $(g + \Gamma)$.

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Received
14 II 1968

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Note: Figure translations are in progress. See original paper for figures.