THE CAUCHY PROBLEM FOR A SECOND-ORDER EQUATION WITH TWO INDEPENDENT VARIABLES IN THE COMPLEX DOMAIN

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Abstract

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THE CAUCHY PROBLEM FOR A SECOND-ORDER EQUATION WITH TWO INDEPENDENT VARIABLES IN THE COMPLEX DOMAIN

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1. In the present paper we consider the solution of the Cauchy problem for a second-order equation with two independent complex variables z_1, z_2. The coefficients of the equation and the initial data are assumed to be analytic functions of z_j, real for real values of z_j $(j = 1, 2)$.

The solution is written in explicit form in terms of the Riemann kernel, whose existence in the complex domain was obtained in the monograph of I. N. Vekua ([1]), to which our work is very closely related (see also the papers of G. Levy—in ([2] their exposition and bibliography are contained). The exposition is carried out in the coordinate-free apparatus of differential forms (see, for example, ([3,4])). At the end of the paper an application of the results to the theory of electrochemical treatment of metals is indicated.

2. We shall consider the equation

$$u_{z_1 z_1} + u_{z_2 z_2} + a u_{z_1} + b u_{z_2} + cu = f, \tag{1}$$

where a, b, c, f are holomorphic functions of z_1, z_2; $z_j = x_j + iy_j$ $(j = 1, 2)$; x_j, y_j are real variables.

We shall assume that on the complex curve $F : F(z_1, z_2) = 0$ (this is a real two-dimensional manifold in a 4-dimensional real space) initial data are prescribed, i.e., the values u and du.

Introduce in the two-dimensional complex space $C^2 = \{z_1, z_2\}$ the metric

$$ds^2 = \frac{1}{2}\left(dz_1^2 + dz_2^2 + \overline{dz}_1^2 + \overline{dz}_2^2\right). \tag{2}$$

The operator of metric conjugation $*$ in the metric (2) has the form:

$$* \, dz_1 = \overline{dz}_1 \, dz_2 \, \overline{dz}_2 \quad \text{etc.}; \qquad * \; dz_1 \, \overline{dz}_1 = dz_2 \, \overline{dz}_2 \quad \text{etc.}$$

The Laplace operator $\Delta = d\delta + \delta d$ (see ([3])) for our metric (2) has, on 0-forms (i.e., on functions on C^2), the form

$$\Delta = \delta d = \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} + \frac{\partial^2}{\partial \overline{z}_1^2} + \frac{\partial^2}{\partial \overline{z}_2^2}.$$

In what follows we shall deal only with analytic forms (see ([4])), for which our 4-dimensional apparatus allows a certain simplification. Namely, introduce the two-dimensional "holomorphic" conjugation

$$\circledast\omega = - * \left(\omega \, d\overline{z}_1 \, d\overline{z}_2\right).$$

On holomorphic forms $\circledast$ acts as if z_1, z_2 were ordinary real Euclidean variables:

$$\circledast f(z_1, z_2) = f(z_1, z_2) \, dz_1 \, dz_2, \qquad \circledast(a \, dz_1 + b \, dz_2) = a \, dz_2 - b \, dz_1,$$

$$\circledast(v \, dz_1 \, dz_2) = v(z_1, z_2).$$

This remark makes it possible to use the formal analogy that exists between the real and complex Riemann formulas. For holomorphic functions u, v in C^2 Green' s formula holds (see ([3])):

$$\circledast v \cdot \Delta u - \circledast u \cdot \Delta v = d\left(v \cdot \circledast du - u \cdot \circledast dv\right). \tag{3}$$

We rewrite (1) in invariant form

$$Lu \equiv \Delta u + \circledast(du \cdot A) + cu = f, \tag{4}$$

where $A = a \, dz_2 - b \, dz_1 = \circledast(a \, dz_1 + b \, dz_2)$.

Introduce the operator $L\circledast$, formally adjoint to the operator L, by the formula

$$L \circledast v = \Delta v - \circledast(Av) + cv,$$

where v is a holomorphic function in C^2. From (3) it follows that

$$\circledast v \cdot Lu - \circledast u \cdot L \circledast v = d(v \cdot \circledast du - u \cdot \circledast dv + uv \cdot A). \tag{5}$$

Fig. 1

Figure 1: Fig. 1

Integrating (5) over an oriented two-dimensional domain D with boundary bD and using Stokes' formula (3), we obtain

$$\iint_D (\circledast v \cdot Lu - \circledast u \cdot L \circledast v) = \int_{bD} v \cdot \circledast du - u \cdot \circledast dv + uvA. \tag{6}$$

Fig. 1

If v is a known solution of the equation $L \circledast v = 0$, then the left-hand side in (6) becomes the known quantity

$$\iint_D \circledast v \cdot f.$$

Let now the domain D be a curvilinear triangle with a right angle at the point $Z = (Z_1, Z_2)$, with legs Γ_1 and Γ_2 going along the characteristics $\zeta_1 = Z_1$ and $\zeta_2 = Z_2$, and with hypotenuse Γ, which joins on the curve F of the initial data the points A_1 and A_2 where Γ_1 and Γ_2 intersect F (see Fig. 1)*. Then the right-hand side of (6) is represented as the sum

$$\int_{bD} = \int_{\Gamma_1} + \int_{\Gamma_2} + \int_{\Gamma}.$$

Using the characteristic property of Γ_j and Stokes' formula for Γ_j, we have

$$\int_{\Gamma_1} v \circledast du = -i \int_{\Gamma_1} v\, du = -iuv \,\big|_{b\Gamma_1} + i \int_{\Gamma_1} u\, dv =$$

$$= -i\big[(uv)(Z) - (uv)(A_1)\big] + i \int_{\Gamma_1} u\, dv$$

and, similarly,

$$\int_{\Gamma_2} v \circledast du = i\big[(uv)(A_2) - (uv)(Z)\big] - i \int_{\Gamma_2} u\, dv,$$

$$\int_{\Gamma_1} u \circledast dv = -i \int_{\Gamma_1} u\, dv; \qquad \int_{\Gamma_2} u \circledast dv = i \int_{\Gamma_2} u\, dv.$$

Now (6) can be rewritten in the form

$$\iint_D f \circledast v = -2i(uv)(Z) + i\big[(uv)(A_1) + (uv)(A_2)\big] + \int_{\Gamma_1} (2i\,dv + A)u +$$

$$+ \int_{\Gamma_2} (-2i\,dv + A)u + \int_{\Gamma} (v \circledast du - u \circledast dv + Auv).$$

* Fig. 1 is schematic, since it depicts the situation in the 4-dimensional (real) space C^2. The surfaces $\zeta_1 = Z_1$ and $\zeta_2 = Z_2$ are two-dimensional; they intersect the two-dimensional surface F at the points A_1 and A_2. The curve Γ joins these points, passing along F.

To obtain the Riemann formula it is enough to require that the function v satisfy ordinary differential equations on the characteristics

$$2i\,dv + A = 0 \text{ on } \Gamma_1(\zeta_1 = Z_1); \qquad -2i\,dv + A = 0 \text{ on } \Gamma_2(\zeta_2 = Z_2)$$

with the initial condition $v(Z) = 1$. The existence of such a function $v(Z, \zeta)$ has been proved for the case of complex arguments in ([1]).

Finally we have the Riemann formula

$$\begin{aligned} u(Z) = &-\frac{1}{2i}\iint_D f \circledast v + \frac{1}{2}\big[(uv)(A_1) + (uv)(A_2)\big] \\ &+ \frac{1}{2i}\int_\Gamma (v \circledast du - u \circledast dv + Auv), \end{aligned} \tag{7}$$

which gives an explicit solution at the point $Z = (Z_1, Z_2)$ in terms of the Riemann kernel $v(Z, \zeta)$, the right-hand side f of equation (4), and the initial data u and du on the curve $\Gamma \subset F$.

3. In ([5,6]) it was shown that the mathematical description of electrochemical machining of a metal leads to the necessity of solving the Cauchy problem for the Laplace equation with analytic initial data. The two-dimensional planar case (it is analyzed in detail with examples in ([6])) leads to the equation $u_{z_1 z_1} + u_{z_2 z_2} \equiv u_{\zeta_1 \zeta_2} = 0$ and to the Riemann function $v(Z, \zeta) = 1$.

The axisymmetric problem leads to the Euler equation

$$u_{z_1 z_1} + u_{z_2 z_2} + \frac{1}{z_1} u_{z_1} = 0$$

with initial data even in z_1. The Riemann kernel in this case is expressed through the Gaussian hypergeometric function (see ([1,7]))

$$v(Z_1, Z_2; \zeta_1, \zeta_2) = (\zeta_2 - Z_1)^{i/2}(Z_2 - \zeta_1)^{-i/2} F\left(-\frac{i}{2}, \frac{i}{2}; 1; \frac{(Z_1 - \zeta_1)(Z_2 - \zeta_2)}{(Z_1 - \zeta_2)(Z_2 - \zeta_1)}\right).$$

Let us note, in conclusion, that the purpose of our exposition is only to carry out preliminary and formal considerations. We have not at all touched upon more complicated questions, such as the geometric complications that arise in our construction of formula (7). (Obviously, our construction is valid only locally, which, however, is sufficient for most electrochemical applications.)

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Note: Figure translations are in progress. See original paper for figures.