ON MINKOWSKI’ S PROBLEM CONCERNING A SYSTEM OF INHOMOGENEOUS LINEAR FORMS

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Abstract

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ON MINKOWSKI' S PROBLEM CONCERNING A SYSTEM OF INHOMOGENEOUS LINEAR FORMS

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Let K, φ, Δ, E be unimodular matrices of dimension n, where

$$K = \begin{pmatrix} K_1 & & & 0 \\ & K_2 & & \\ & & & \ddots \\ 0 & & & K_n \end{pmatrix}, \qquad \varphi^T = \varphi^{-1}, \qquad \Delta = \begin{pmatrix} \varepsilon_{11} & & & 0 \\ \varepsilon_{21} & \varepsilon_{22} & & \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \varepsilon_{n1} & \varepsilon_{n2} & \cdots & \varepsilon_{nn} \end{pmatrix},$$

E is an integral unimodular matrix.

Theorem 1. *Let the matrix A be such that*

$$A = K\varphi\Delta E.$$

Consider the system of inhomogeneous forms

$$A(X) + \alpha,$$

where X is an integral vector and α is an arbitrary vector. Put

$$\begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ y_n \end{pmatrix} = A(X) + \alpha, \qquad L(A, \alpha) = |y_1 y_2 \cdots y_n|;$$

then

$$\min_X L(A,\alpha) \leq \left(\sum_1^n \varepsilon_{ii}^2/n\right)^{n/2} \frac{1}{2^n}.$$

Let us note that if $\varepsilon_{11}, \varepsilon_{22}, \dots, \varepsilon_{nn}$ are all equal to one, then Minkowski's assertion is true for the matrix A.

Theorem 2. *Any unimodular matrix A of dimension $n = 2, 3$ can be represented in the form of a product $A = K\varphi\Delta E$, where K is diagonal, φ is orthogonal, Δ is triangular with $\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 1$, and E is an integral unimodular matrix.*

The proof of Theorem 2 is based on the following considerations: $A = K\varphi\Delta E$ is equivalent to $K^{-1}A = \varphi\Delta E$, and this is equivalent to

$$(K^{-1}A)^T(K^{-1}A) = (\Delta E)^T(\Delta E);$$

the latter equality reduces to the question of equivalence of binary (ternary) quadratic forms.

Namely, consider the forms: $F_1 = X^T((K^{-1}A)^T(K^{-1}A))X$ and $F_2 = X^T(\Delta^T\Delta)X$, where K, Δ are diagonal and triangular matrices, and A is a fixed unimodular matrix. It is shown that for any unimodular A one can choose K and Δ ($\varepsilon_{11} = \varepsilon_{22} = \varepsilon_{33} = 1$) so that $F_1 \sim F_2$, and this is equivalent to the assertion of Theorem 2.

It should be noted that in the proof of Theorems 1 and 2 the Minkowski theorem on a centrally symmetric convex body was not used. For example, in Theorem 2, instead of a centrally symmetric convex body, the following obvious fact is used: if $A = K\varphi\Delta E$, then $(A^{-1})^T = K'\varphi'\Delta'E'$.

We note that Theorem 2, apart from its application to the Minkowski problem, is of independent interest. It is easy to see that Theorem 2 implies: for any unimodular matrix, up to a hyperbolic rotation, the unit cube (the cube with edge length one) is a factor space $\bmod \Lambda$ of the lattice Λ generated by the matrix A.

In conclusion we note that such an approach to the inhomogeneous Minkowski problem seems to us new and promising.

This article arose on the basis of the work of the seminar on the geometry of numbers of Samarkand State University named after Alisher Navoi.

Samarkand State University
named after A. Navoi

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Note: Figure translations are in progress. See original paper for figures.

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