ON THE STRUCTURE OF SOME COMPLETE SURFACES OF NONPOSITIVE CURVATURE

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Abstract

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MATHEMATICS

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ON THE STRUCTURE OF SOME COMPLETE SURFACES OF NONPOSITIVE CURVATURE

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1. Let an unclosed complete two-dimensional metric manifold $\Phi \in C^2$ be immersed in three-dimensional Euclidean space R^3, i.e. let Φ be a surface of class C^2 in R^3, complete in the intrinsic sense. This surface may be incomplete in the extrinsic sense, and may have self-intersections and self-overlaps. In R^3 introduce Cartesian coordinates x, y, z, and denote by $z(u)$ the z-coordinate of the point $u \in \Phi$.

Let S be the unit sphere in R^3 with center at the point $O = (0,0,0)$. By u' denote the pair of ends of the diameter of the sphere S perpendicular to the tangent plane to Φ at the point u, and by $\alpha(u)$ the smaller angle between this diameter and the z-axis. Let

$$\Phi^+(h) = \{u \in \Phi,\ z(u) > h\}, \qquad \Phi^-(h) = \{u \in \Phi,\ z(u) < h\},$$

and

$$L(h) = \{u \in \Phi,\ z(u) = h\}.$$

Curves on Φ orthogonal to the level lines $L(h)$ will be called **gradient lines**. We shall say that a sequence of points diverges on Φ if it contains no compact subsequence. Let a sequence of points $u_n \in \Phi$ diverge on Φ, and let $v \in S$ be a limiting point of the set $\{u'_n\}$. To the set $\Gamma \subset S$ we assign all such points v for all possible sequences $\{u_n\}$ diverging on Φ. The set Γ is closed and symmetric on S. By Φ'_0 denote the set of those u' to which correspond points $u \in \Phi$ of zero Gaussian curvature. The set Φ'_0 has measure zero on S.

2. Theorem 1. *Let the z-axis not intersect Φ'_0 and Γ (if $\Gamma \neq S$, one can always choose such a direction of the z-axis). Then Φ has finite connectivity, and the sets $L(h)$, outside some set bounded on Φ, will consist of one and the same, for all h, even number $2m$ of simple unbounded arcs. Moreover, on Φ there will be a finite number p of elliptic points and a finite number q of hyperbolic points at which $\alpha(u) = 0$. The equality*

$$p - q + m = \chi(\Phi), \tag{1}$$

holds, where $\chi(\Phi)$ *is the Euler characteristic of the manifold* Φ*.*

In the proof of Theorem 1 the following two lemmas are used:

Lemma 1. *Let* H *be an unbounded-on-*Φ *component of the set* $\Phi^+(h)$*, and let the domain* G *be a part of the domain* H*, also unbounded on* Φ*. Let* g *be the boundary of* G*, and let the set* $g \cap H$ *be bounded on* Φ*. Then, if*

$$\sup_{u \in G} z(u) < +\infty,$$

then

$$\inf_{u \in G} \alpha(u) = 0.$$

Lemma 2. *Let* G *be a component of the set* $\Phi^+(h_0)$ *for which*

$$\inf_{u \in G} \alpha(u) > 0.$$

Then the boundary of the component G *consists of one simple curve, and all sets* $G \cap L(h)$*,* $h > h_0$*, are homeomorphic to this curve.*

The proofs of Lemmas 1 and 2 are based on the fact that: 1) in a neighborhood of any point $u \in \Phi$ where $\alpha(u) \neq 0$, the set $L(h)$ will be a simple arc, and 2) along any gradient line γ the equality

$$dz(u) = \sin \alpha(u)\, ds(u)$$

holds, where $s(u)$ is the natural parameter on γ.

Plan of the proof of Theorem 1. From the hypothesis of the theorem it follows that there exists an $a_0 > 0$ for which the set $\Phi_1 = \{u \in \Phi,\ a(u) \leq a_0\}$ is compact in Φ, and the set $\Phi_2 = \{u \in \Phi,\ a(u) = 0\}$ consists of a finite number of points of nonzero Gaussian curvature: p elliptic points $u_1, \dots, u_p$ and q hyperbolic points $u_{p+1}, \dots, u_{p+q}$. We may assume that $z(u) \in [0, 1]$ for $u \in \Phi_1$. Then for $h \geq 1$ ($h \leq 0$) all $L(h)$ are homeomorphic to $L(1)$ ($L(0)$). If $p = q = 0$, then Φ is homeomorphic to the plane or to an infinite circular cylinder. If $p + q > 0$ and there is a gradient line going from a point $u \in L(0) \cup L(1)$ to the point u_i, $i \leq p$, then $p = 1$, $q = 0$, and Φ is homeomorphic to the plane. If there is no such line, then from each component $l \in L(1)$ ($l \in L(0)$) there issues a gradient line going to a point u_i, $i > p$. Since exactly two gradient lines enter each hyperbolic point u_i from above (from below), the number of components in $L(1)$ ($L(0)$) is not greater than $2q$, and Φ is finitely connected. Then, by a special construction, one can cut out from Φ a compact part Φ_3 and then complete it to a closed manifold $\overline{\Phi}$, extending the function $z(u)$ to $\overline{\Phi}$ in such a way that applying the known Morse equality to $z(u)$ on $\overline{\Phi}$ leads to (1).

3. A **tube** T is a two-dimensional noncompact doubly connected metric manifold with boundary homeomorphic to a circle. Just as in § 1, by means of sequences of points diverging on T, we define, for a smooth tube T immersed in R^3, a set $\Gamma(T) \subset S$ analogous to the set Γ for Φ. With the help of Lemmas 1 and 2 one can prove

Theorem 2. *If, for a tube T of class C^2, the set $\Gamma(T) \neq S$, then T is unbounded in space in the direction of any diameter of the sphere S not intersecting the set $\Gamma(T)$. In particular, if T has a finite-sheeted spherical image, then $\Gamma(T) \neq S$.*

4. Everywhere below we shall assume that the surface Φ, defined in § 1, has a one-to-one spherical image. Then the Gaussian curvature K on Φ does not change sign (1). If $K \geq 0$, then, as proved in (2), Φ will be a complete convex surface. We consider only the case when $K \leq 0$. Such a surface Φ will be called a **simplest hyperbolic surface**.

Theorem 3. *If Φ is a simplest hyperbolic surface, then Φ is homeomorphic to the plane or to an infinite circular cylinder, i.e. $\chi(\Phi) = 1$ or $\chi(\Phi) = 0$.*

In the case under consideration it follows easily from Theorem 1 that $\chi(\Phi) \geq -2$, i.e. Φ goes to infinity by no more than four tubes. These tubes may narrow, i.e. be so-called horns, or widen, i.e. be cups (for the definitions of a horn and a cup see (3) or (1)). The cases $\chi(\Phi) = -1$ and $\chi(\Phi) = -2$ are excluded with the help of the following two lemmas:

Lemma 3. *If L is a closed geodesic on Φ, then its spherical image divides the sphere S into two regions of equal area.*

Lemma 3 is a generalization of the well-known theorem of Jacobi on the indicatrix of the principal normals of a closed curve.

Lemma 4. *If the tube T is a horn of the surface Φ, then the part of the boundary of the spherical image of the surface $\overline{\Phi}$ corresponding to T will be a great circle on S.*

It follows from Theorem 2 that the horn considered in Lemma 4 is unbounded in space, and for such a horn Lemma 4 was proved in (1).

From Lemma 3 it follows that there cannot be two nonintersecting closed geodesics on Φ, and from Lemma 4 it follows that there can be at most one horn on Φ.

5. From the point of view of metric properties, the simplest hyperbolic surfaces split into three classes: 1) $\chi(\Phi) = 1$ –homeomorphic to the plane: example $z = xy$; 2) $\chi(\Phi) = 0$ and Φ has two cups: example

$x^2 + y^2 - z^2 = 1$; 3) $\chi(\Phi) = 0$ and Φ has a horn: an example is $z = \exp\left(\dfrac{1}{x^2 + y^2}\right)$.

Theorems 1 and 3 make it possible to give a complete classification of the topological structure of plane sections of the simplest hyperbolic surfaces, depending on the position of the spherical image of the cutting plane relative to the spherical image of the surface itself.

Theorem 4. *If the simplest hyperbolic surface $\Phi \in C^2$ has a horn, then it is projected one-to-one onto the plane in the direction of the horn. Its projection will be the entire plane, with the exception of some bounded closed convex set.*

The direction of the horn is defined as the direction of the ray to which the segments joining any point X to a point Y on the horn converge when Y recedes along the horn. The existence of this ray was proved in (^{1}).

6. *The simplest hyperbolic surface need not be complete in the external sense.* Such, for example, is the surface

$$F_1 : \ z = x \operatorname{tg} y - \exp(\operatorname{tg}^2 y) \cdot \sin \operatorname{tg} y,$$

where $x \in (-\infty, +\infty)$, and $y \in (-\pi/2, +\pi/2)$. The surface F_1 will be complete in the internal sense, but the points of the planes $y = \pm\pi/2$ will be limit points for sequences of points diverging on F_1. One can give a number of conditions for completeness, in the external sense, of the simplest hyperbolic surface. Among them we note that, by virtue of Theorem 4, *the simplest hyperbolic surface having a horn will be complete in the external sense.*

Below we consider only the simplest hyperbolic surfaces that are complete in the external sense. Let F be such a surface of class C^2, and let F^* be its spherical image. We shall also assume that F has no self-intersections outside some compact domain on F.

Theorem 5. *Let F be a doubly connected simplest hyperbolic surface defined above. Then the set $S \setminus \overline{F^*}$ consists of two convex domains (they may also be empty; $\overline{F^*}$ is the closure of F^*). If F has a horn, then the part of the set $S \setminus F^*$ corresponding to the horn will be a closed hemisphere.*

Let a tube $T \subset F$ be given. By T_k (F_k) we denote the surface obtained from T (F) by a homothety with coefficient $k > 0$ and with fixed center. The **set of limiting rays** of the surface F (the tube T) is the set

$$A(F) = \lim_{k\to 0} F_k (A(T) = \lim_{k\to 0} T_k).$$

If $A(T)$ is a surface (a cone), then $A(T)$ is called the **limiting cone** of the tube T; degeneracies of the limiting cone are allowed. If all tubes of the surface F have a limiting cone, then $A(F)$ is called the limiting cone of the surface F (if F is simply connected, then one may regard F as having one tube, since, by cutting from F a domain homeomorphic to a disk, we obtain a tube). *There exist simplest hyperbolic surfaces that do not have a limiting cone*: for example, for the surface

$$F_2 : \ z = y \operatorname{arc}\operatorname{tg} x + \exp x^2 \cdot \sin x,$$

where $x \in (-\infty, +\infty)$, $y \in (-\infty, +\infty)$, the set $A(F)$ will be the whole space.

The following theorems have been proved:

Theorem 6. *If a doubly connected simplest hyperbolic surface F has a limiting cone $A(F)$, then $A(F)$ consists of two convex cones. The simplest hyperbolic surface with a horn always has a limiting cone consisting of a convex cone and a ray whose continuation goes inside this cone.*

Theorem 7. *If a simply connected simplest hyperbolic surface F is given by the equation $z = f(x, y)$ on the whole plane x, y and has a limiting cone $A(F)$, then $A(F)$ will be a hyperbolic cone with a one-to-one spherical image.*

Hyperbolic cones with one-to-one spherical image were studied in (^{4}) (the case of a polyhedral angle was analyzed in (^{5})). Such a cone consists, if one takes the nondegenerate case, of four convex

parts, changing the direction of convexity four times when going around the vertex of the cone, and has as its spherical image a domain "of astroid type," whose boundary consists of four convex arcs directed inward into the domain. Any plane passing through the vertex of this cone intersects it in no more than four components.

The proof of Theorems 6 and 7 makes essential use of the results mentioned in §5 on the topological structure of plane sections of the simplest hyperbolic surfaces. Roughly speaking, they consist in the fact that in a plane section of the cup of a doubly connected simplest hyperbolic surface there can be no more than two branches going off to infinity, while in a plane section of the surface considered in Theorem 7 there can be no more than four branches going off to infinity.

Let a doubly connected simplest hyperbolic surface F have the limiting cone $A(F)$. By Theorem 6, it consists of two convex cones K_1 and K_2. Let the open convex domains K_1^* and K_2^* be the sets of interior points of the spherical images of the cones K_1 and K_2, respectively. If one takes into account the structure of the plane sections of $A(F)$ and F, then it is natural to call the set $A^*(F) = S \setminus (K_1^* \cup K_2^*)$ the spherical image of the cone $A(F)$. For the limiting cone $A(F)$ of the surface F considered in Theorem 7, the spherical image $A^*(F)$ was introduced in (4).

Theorem 8. *If the simplest hyperbolic surface F, considered in Theorem 6 or 7, has a limiting cone $A(F)$, then $A^*(F) = \overline{F^*}$. For the simplest hyperbolic surface with a rim this always holds.*

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References

1. A. L. Verner, *Vestnik LGU*, **7**, 16 (1965).
2. A. V. Pogorelov, *Surfaces of Bounded Extrinsic Curvature*, Kharkov, 1956.

3. S. E. Cohn-Vossen, *Some Questions of Differential Geometry in the Large*, Moscow, 1959.

4. A. L. Verner, *Uch. Zap. Leningrad State Pedagogical Institute named after A. I. Herzen*, **328**, 44 (1967).

5. A. L. Verner, *Matem. Sbornik*, **71** (113), no. 2, 261 (1966).

Note: Figure translations are in progress. See original paper for figures.

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