ON CERTAIN PROPERTIES OF REPRESENTATIONS OF AN INFINITE-DIMENSIONAL CLIFFORD ALGEBRA

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Abstract

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V. Ya. GOLODETS

ON CERTAIN PROPERTIES OF REPRESENTATIONS OF AN INFINITE-DIMENSIONAL CLIFFORD ALGEBRA

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A **representation of a Clifford algebra** is a set of linear self-adjoint bounded operators $\{A_k\}_1^\infty$, acting in a separable Hilbert space H and satisfying the commutation rules

$$A_k^2 = I, \qquad A_{kA}l + A_{lA}k = 0 \quad (k \neq l) \quad (k, l = 1, 2, \ldots). \qquad (1)$$

In the present article we study the structure of an operator commuting with the operators $\{A_k\}_1^\infty$ satisfying relations (1). We also give a necessary and sufficient condition for the weakly closed ring M, generated by the operators $\{A_k\}_1^\infty$, to be a factor of type II_1 in the sense of Murray and von Neumann.

1. Let Γ be the space of binary sequences $x = x_1 x_2 \ldots$ ($x_i = 0$ or 1). Under componentwise addition modulo 2, Γ becomes a group. Denote by Δ the subgroup of Γ whose generators are the sequences δ_k, in which the k-th component is equal to 1, and the remaining ones to zero. We single out the σ-algebra S of subsets of Γ with generators $\Gamma_k = \Gamma(x : x_k = 0)$ and $\Gamma'_k = \Gamma(x : x_k = 1)$ $(k = 1, 2, \ldots)$. A measure μ, defined on the subsets S, is called **quasi-invariant** if the measures $\mu(x)$ and $\mu(x+\delta)$, for all $\delta \in \Delta$, are equivalent. A quasi-invariant measure μ is called **ergodic** if an arbitrary bounded measurable function $f(x)$ on Γ, satisfying $f(x+\delta) = f(x)$ for almost all $x \in \Gamma$ and all $\delta \in \Delta$, is constant on a set of measure one.

For convenience, put

$$A_k = A_{2k-1}, \qquad B_k = A_{2k} \quad (k = 1, 2, \ldots).$$

According to results of L. Gårding and A. Wightman ([1]), if a representation of the Clifford algebra $\{A_k, B_k\}_1^\infty$ is given, then the space H can be realized in the form of a direct integral of Hilbert spaces H_x over Γ with respect to a quasi-invariant measure μ,

$$H = \int_{\Gamma}^{\oplus} H_x \, d\mu(x), \tag{2}$$

where to each element $f \in H$ there corresponds a vector-function $f(x)$ on Γ with values in H_x, and

$$(B_{kA}kf)(x) = i^{-1}(-1)^{x_k} f(x),$$

$$(A_k f)(x) = i_k(x) C_k(x) f(x + \delta_k) \sqrt{\frac{d\mu(x+\delta_k)}{d\mu(x)}} \tag{3}$$

$$(k = 1, 2, ...),$$

where $i_k(x) = (-1)^{x_1 + \cdots + x_k}$, and $\{C_k(x)\}_1^{\infty}$ is a family of measurable operator-valued functions satisfying the functional equations: for almost

for all $x \in \Gamma$:

$$\begin{gathered} C_k(x) C_k^*(x) = I, \qquad C_k^*(x) = C_k(x + \delta_k), \\ C_k(x) C_l(x + \delta_k) = C_l(x) C_k(x + \delta_l) \qquad (k \neq l) \\ (k, l = 1, 2, ...). \end{gathered} \tag{4}$$

Let T be a self-adjoint bounded operator commuting with the operators of the representation $\{A_k, B_k\}_1^{\infty}$. From commutativity with $B_{kA}k$ $(k = 1, 2, ...)$ it follows that T can be represented in the form of an operator-measurable bounded function $T(x)$ $(T(x) = T^*(x)$ for almost all $x \in \Gamma)$

$$(Tf)(x) = T(x) f(x),$$

and from commutativity with the operators A_k $(k = 1, 2, ...)$ it follows that for almost all x

$$C_k^*(x) T(x) C_k(x) = T(x + \delta_k) \qquad (k = 1, 2, ...). \tag{5}$$

Theorem 1. *Let a representation of the Clifford algebra $\{A_k, B_k\}_1^{\infty}$ be given, for which the quasi-invariant measure μ is ergodic. If $T = \{T(x)\}$ is a bounded self-adjoint operator commuting with the operators of the representation, then there exists a measurable mapping $U(x)$ of the set Γ into the unitary operators of the spaces H_x, for which*

$$T(x) = U(x) C U^*(x)$$

for almost all x, where C is a bounded self-adjoint operator independent of x.

We note the most essential point of the proof. Denote by σ the measure belonging to the maximal spectral type of the operator T. Consider the weakly closed commutative $*$-ring generated by the operators $B_{kA}k$ $(k = 1, 2, \dots)$ and T. It is known that there exists a certain self-adjoint operator R which generates this commutative ring. Denote by ρ the measure belonging to the maximal spectral type of R. It turns out that ρ is equivalent to $\mu \times \sigma$ $(\rho \sim \mu \times \sigma)$. From this result Theorem 1 follows easily.

As a consequence of Theorem 1 we obtain:

Every quasi-invariant ergodic measure on Γ with respect to the group Δ is equivalent to the direct product of two quasi-invariant ergodic measures on $\hat{\Gamma}$.

2. Theorem 2. *Let a representation of the Clifford algebra $\{A_k\}_1^\infty$ be given. Denote by M the minimal weakly closed ring generated by the operators $\{A_k\}_1^\infty$. Suppose that M is a factor in the sense of Murray and von Neumann. In order that M have type II_1, it is necessary and sufficient that on the elements of M one can define a linear homogeneous positive functional T, continuous with respect to the weak topology in M, for which*

$$T(I) = 1, \qquad T(A_{i_1} \dots A_{i_k}) = 0, \tag{6}$$

where $i_1 < \dots < i_k$; $i_s = 1, 2, \dots$ $(s = 1, \dots, k)$; $k = 1, 2, \dots$.

Necessity is obvious. Let us dwell on the proof of sufficiency. According to the results stated in (5), it is enough to show that:

1) $T(C_1C_2) = T(C_2C_1)$ for $C_i \in M$ $(i = 1, 2)$;

2) if P is a projection operator in M and $T(P) = 0$, then $P = 0$.

Suppose that the operators C_i $(i = 1, 2)$ can be represented in the form of a finite polynomial in the A_k $(k = 1, 2, \dots)$. Select all operators A_k which enter into the expansion for C_1 and C_2. Construct the finite-dimensional Clifford algebra M_1, whose generators are these operators. It is well known (4) that M_1 is algebraically isomorphic to the full matrix algebra. By the uniqueness of the trace in M_1, it coincides with T; from the properties of the trace it follows that

$$T(C_2C_1) = T(C_1C_2). \tag{7}$$

For arbitrary operators C_1, C_2 from M, (7) remains valid by virtue of the continuity of the functional T.

Let us turn to the consideration of property 2). Suppose that T does not have this property; then M cannot have type II_1. Denote by N the set of all elements A of M for which $T(A^*A) = 0$. Obviously, N is a two-sided closed ideal in M.

By assumption, N is nonempty. Since a factor of type III is simple, the only remaining possibility is to suppose that M has either type II_∞ or type I_∞.

Form the space of cosets M/N. If $A \in M$, but $A \notin N$, then by $\widetilde{A}$ we denote the coset that contains the element A. Define in the linear space M/N the scalar product

$$\langle \widetilde{A}, \widetilde{B} \rangle = T(AB^*),$$

where $A \in \widetilde{A}$, $B \in \widetilde{B}$. Obviously, $\langle \widetilde{A}, \widetilde{B} \rangle$ does not depend on the choice of the representatives A and B.

Thanks to the scalar product $\langle \cdot, \cdot \rangle$, the space M/N becomes a pre-Hilbert space; denote its completion by $\widehat{H}$. Define in $\widehat{H}$ an involution J, putting $J\widetilde{A} = \widetilde{A^*}$ for elements of M/N, and on the remaining elements of $\widehat{H}$ define the operator J by continuity. Then $(\widehat{H}, J, M/N)$ is a Hilbert algebra in the sense of I. Segal ([2]).

Denote by L the minimal weakly closed ring generated by the operators $L_{\widetilde{A}}$:

$$L_{\widetilde{A}}\widetilde{X} = \widetilde{A}\widetilde{X} \qquad (\widetilde{A}, \widetilde{X} \in M/N).$$

I. Segal showed that the ring L possesses a measure m on projections, and if a projection has the form $L_{\widetilde{C}}$, where $\widetilde{C} \in M/N$, then $m(L_{\widetilde{C}}) = \langle \widetilde{C}, \widetilde{C} \rangle = T(C^*C)$.

Consider in M the projections $P_1 = \frac{1}{2}(I + iA_1A_2)$ and $P_2 = \frac{1}{2}(I - iA_1A_2)$. Obviously, $P_1 + P_2 = I$ and $P_2 = A_2P_1A_2$. Since, by assumption, M has type either II_∞ or I_∞, P_1 is an infinite projection; moreover, there exists a partially isometric operator U in M for which

$$UU^* = I, \qquad U^*U = P_1.$$

Consequently,

$$I = L_{\widetilde{U}\widetilde{U^*}} = L_{\widetilde{U^*}}L_{\widetilde{U^*}} = L_{\widetilde{U}} \cdot L^*_{\widetilde{U}}, \qquad L_{P_1} = L^*_{\widetilde{U}}L_{\widetilde{U}}. \tag{8}$$

In the cited work, I. Segal proved that if V is a partially isometric operator belonging to some weakly closed ring R with a measure m' on projections, then $m'(V^*V) = m'(VV^*)$. Therefore, in our case (see (6))

$$m(L_{P_1}) = m(L_I) = T(I) = 1;$$

on the other hand (see (6)),

$$m(L_{P_1}) = T(P_1) = T\left(\tfrac{1}{2}(I + iA_1A_2)\right) = \tfrac{1}{2}.$$

The contradiction obtained proves that M cannot have type II_∞ or I_∞.

Thus, if T does not have property 2), then the factor M belongs neither to type I_∞, nor to II_∞, nor to III; but this contradicts the Murray–von Neumann theory. Thus, property 2) is proved.

Theorem 2 shows that a factor representation of type II_1 for the Clifford algebra preserves important properties of the finite-dimensional Clifford algebra. The theorem also gives a simple method for establishing the type of the factor.

Physical-Technical Institute
of Low Temperatures
Academy of Sciences of the Ukrainian SSR

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Note: Figure translations are in progress. See original paper for figures.