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ON RECURSIVE SUBSETS OF SETS OF RECURSIVE FUNCTIONS

MATHEMATICS

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Abstract

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MATHEMATICS

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ON RECURSIVE SUBSETS OF SETS OF RECURSIVE FUNCTIONS

(Presented by Academician P. S. Novikov on 24 IV 1968)

Let χ denote Kleene's numbering of the set $A^{(1)}_{4p}$ of all one-place partial recursive functions (p.r.f.), and let $K(n, x)$ denote Kleene' s universal function ([1]).

Let $G \subseteq A^{(1)}_{4p}$ be some family of p.r.f. We shall call a set $A \subseteq G$ **recursive relative to** G ([2]) if there exists a p.r.f. $g(x)$ such that

$$g(x) = \begin{cases} 0, & \text{if } \chi x \in A, \\ 1, & \text{if } \chi x \in G \setminus A. \end{cases}$$

If $G = A^{(1)}_{4p}$, then Rice's well-known theorem asserts that only two trivial subsets of G (G itself and the empty set) will be recursive.

By $A^{(1)}_{\text{pr}}$, $A^{(1)}_{\text{or}}$ we shall denote respectively the sets of all one-place primitive-recursive functions (p.r.f.) and all one-place general recursive functions (g.r.f.).

In ([3]) it was observed that if $G = A^{(1)}_{\text{pr}}$ or $G = A^{(1)}_{\text{or}}$, then the set of functions

$$A^{y_1,\dots,y_k}_P = \{f \in G \mid P(f(y_1), \dots, f(y_2))\}$$

($P(x_1, \dots, x_k)$ is a recursive predicate; $y_1, \dots, y_k$ is a sequence of natural numbers) is a recursive subset relative to G.

In the present note a broader family of subsets recursive relative to G is constructed than the one indicated above; in the case where G is a computable family of g.r.f., a description is obtained of all subsets recursive relative to G. The note also gives a description of all subalgebras of the algebra of p.r.f. $\mathfrak{A}_{\text{pr}} = \langle A^{(1)}_{\text{pr}};\ +,\ *,\ i\rangle$ that are recursive relative to $A^{(1)}_{\text{pr}}$, where the symbols $+$, $*$, i denote respectively the operations of addition of two functions, superposition of two functions, and iteration of a function.

1. By a tuple we shall mean a tuple (possibly empty) consisting of natural numbers $N = \{0, 1, 2, \dots\}$.

With each tuple $a = (i_0, \dots, i_n)$ and set of g.r.f. G we associate the set of functions

$$T_a^G = \{f(x) \mid f(x) \in G \wedge f(0) = i_0 \wedge \dots \wedge f(n) = i_n\}.$$

Let G be a family of g.r.f. such that for each tuple $(i_0, \dots, i_n)$ there is a function $g(x) \in G$ for which $g(s) = i_s$ $(0 \le s \le n)$.

It is not difficult to show that if $\{\alpha_i\}$ and $\{\beta_i\}$ are recursively enumerable sequences of tuples such that

$$A = \bigcup_{i=0}^{\infty} T_{\alpha_i}^G \quad \text{and} \quad G \setminus A = \bigcup_{i=0}^{\infty} T_{\beta_i}^G,$$

then the set A is recursive relative to G.

The following example shows that there exist families of functions $A \subseteq G$ that are recursive relative to G and do not have the form $A_P^{y_1,\dots,y_k}$.

Let us divide all tuples of odd length into two classes Γ_1, Γ_2 as follows:

1) $(a_0, \dots, a_{2t}) \in \Gamma_1 \leftrightarrow a_0 = \dots = a_{2t} = 2t$;
2) $(a_0, \dots, a_{2t}) \in \Gamma_2 \leftrightarrow (\forall i)(i < 2t \wedge i \text{ even} \to (a_0, \dots, a_i) \in \Gamma_1) \wedge (\exists i)(a_i \text{ odd} \vee a_i < 2t)$.

Obviously, the sets Γ_1, Γ_2 can be represented in the form of recursively enumerable sequences $\{\alpha_i\}$ and $\{\beta_i\}$, respectively. Let

$$A = \bigcup_{i=0}^{\infty} T_{\alpha_i}^G,$$

then

$$G \setminus A = \bigcup_{i=0}^{\infty} T_{\beta_i}^G.$$

It is not hard to see that A is not of the form $A_P^{y_1,\dots,y_k}$.

2. Let G be some family of general recursive functions of one variable. A sequence of functions from G, $f_1(x), f_2(x), \dots$, will be called recursively enumerable if the function $B(i, x) = f_i(x)$ is a general recursive function of two variables. By M we shall denote some infinite recursively enumerable subset of the natural numbers.

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A set of functions $A \subseteq G$ is called $(B,g)_M$-closed ($g(x)$ is some function from G), if it contains the functions

$$f_i(x) = \begin{cases} g(x), & \text{if } 0 \le x < i, \\ B(i,x), & \text{if } x \ge i, \end{cases}$$

for all $i \in M$.

Theorem 1. *If there exists a function $g(x) \in G \setminus A$ and a sequence $\{B(i,x)\}$ ($B(i,x)$ is a general recursive function) of functions from G such that, for some infinite recursively enumerable subset of the natural numbers M, the set A is $(B,g)_M$-closed, then A is not recursive relative to G.*

Theorem 1 is a generalization of Theorem 10 (§ 11) stated in [2].

3. In what follows, by G we shall understand a computable family of general recursive functions, i.e.,

$$G = \{F(0,x), F(1,x), ...\},$$

where $F(n,x)$ is a general recursive function.

Theorem 2. *If the set $A \subseteq G$ is recursive relative to G, then A is a computable family.*

Proof. Since A is recursive relative to G, there exists a partial recursive function $g(x)$ such that

$$g(x) = \begin{cases} 0, & \text{if } \chi x \in A, \\ 1, & \text{if } \chi x \in G \setminus A. \end{cases}$$

For some general recursive function $\psi(x)$ we have

$$F(n,x) = K(\psi(n), x).$$

By N_g denote the set of all solutions of the equation $g(x) = 0$. Clearly, the sets N_g and $\rho\psi$ (the range of values of ψ) are recursively enumerable, whence

$$N_A^\chi = N_g \cap \rho\psi$$

is a recursively enumerable set and

$$N_A^\chi = \{\varphi(0), \varphi(1), ...\}$$

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for a suitable general recursive function φ. Obviously, $K(\varphi(n), x)$ is a universal function for A.

Definition. $\beta(x)$ is a *limit point* for $A \leftrightarrow \forall n \exists f \in A\ \forall x (x \leq n \rightarrow \beta(x) = f(x))$.

Theorem 3. *If there exists a function $\beta(x) \in G \setminus A$ which is a limit point for A, then A is not recursive relative to G.*

Proof. Using Theorem 2, it suffices to consider the case where A is a computable family, i.e.,

$$A = \{H(0, x), H(1, x), ...\}$$

for some general recursive function H.

Define the function $h(n)$ as follows:

$$h(n) = \mu t(H(t, 0) = \beta(0) \wedge ... \wedge H(t, n) = \beta(n)).$$

Since β is a limit point for A, $h(n)$ is a general recursive function.

It is clear that A is $(B, \beta)_N$-closed, where $B(n, x) = H(h(n), x)$, and $N = \{0, 1, 2, ...\}$.

Using Theorem 1, we conclude that A is nonrecursive relative to G. We shall say that sequences $\{\alpha_i\}$ and $\{\beta_i\}$ of tuples α_i, β_i are **effectively separable** if there exist recursively enumerable sequences of tuples α_i' and β_i' such that $\{\alpha_i\} \subseteq \{\alpha_i'\}$, $\{\beta_i\} \subseteq \{\beta_i'\}$, and $\{\alpha_i'\} \cap \{\beta_i'\} = \emptyset$ ($\emptyset$ is the empty set).

Theorem 4. *A set $A \subseteq G$ is recursive relative to G if and only if*

$$A = \bigcup_{i=0}^{\infty} T_{\alpha_i}^G \quad \text{and} \quad G \setminus A = \bigcup_{i=0}^{\infty} T_{\beta_i}^G$$

for some effectively separable sequences $\{\alpha_i\}, \{\beta_i\}$ of tuples.

In the proof of Theorem 4 one uses Theorem 3 of the present note and Theorem 5 (§ 10) from (2).

4. Consider the algebra of p.r.f. $\mathfrak{A}_{\text{pr}} = \langle A_{\text{pr}}^{(1)}; +, *, i \rangle$. Take a word in the alphabet $\Gamma = \{\alpha, \beta\}$ beginning with α, for example $\alpha\beta\alpha$, and define the subalgebra $\mathfrak{A}_{\alpha\beta\alpha}$ of the algebra $\mathfrak{A}_{\text{pr}}$ as follows:

$$\mathfrak{A}_{\alpha\beta\alpha} = \{f(x) \mid f(x) \in A_{\text{pr}}^{(1)} \wedge f(0) = 0 \wedge f(1) \in N \wedge f(2) = 0\}.$$

Theorem 5. *A subalgebra of the algebra $\mathfrak{A}_{\text{pr}}$ is recursive if and only if it has the form $\mathfrak{A}_{a_1, \ldots, a_n}$ for some word $a_1 \ldots a_n$ beginning with α.*

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CITED LITERATURE

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2. V. A. Uspensky, *Lectures on Computable Functions*, 1960.
3. E. A. Polyakov, DAN, 178, No. 2 (1968).

Note: Figure translations are in progress. See original paper for figures.

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