A Numerical Method for Finding Partially Stable Singular Points of Ordinary Differential Equations

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Abstract

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A Numerical Method for Finding Partially Stable Singular Points of Ordinary Differential Equations

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1. In the note ([1]) an algorithm $\mathfrak{A} = \mathfrak{A}(\{A_n\}_1^\infty;\ \{s_n\}_0^\infty;\ \{a_n\}_0^\infty)$ was constructed for control by the numerical Euler method, for finding an asymptotically stable singular point of a system of ordinary differential equations. The study of this algorithm was based on the converse of A. M. Lyapunov' s theorem obtained by I. G. Malkin ([2]). The present note is devoted to extending the algorithm $\mathfrak{A}$ to the case of partial asymptotic stability. We note that partially stable systems arise, in particular, in passing from a stable differential equation of higher order to a system of first-order equations. The justification of convergence of the algorithm $\mathfrak{A}$ under conditions of partial stability requires a corresponding converse of A. M. Lyapunov' s theorem. The results of V. V. Rumyantsev ([3]) and C. Corduneanu ([4]) available in this direction are insufficient for our purposes, since they concern partial stability in the small. In paragraph 2 of the present note, under some additional assumptions, we establish a converse of A. M. Lyapunov' s theorem on stability in the large for the case of partial stability. This, in particular, makes it possible to justify the convergence of the algorithm $\mathfrak{A}$ for finding a saddle point of the Lagrange function $\varphi(x, y)$ for a problem of concave programming (when the function $\varphi(x, y)$ is strictly concave in x and linear in y).

2. Let E_m and E_{n-m} be Euclidean spaces. Consider the system of equations

$$dx/dt = f(t, x, y), \qquad dy/dt = g(t, x, y), \tag{1}$$

where the vector x and the vector-function $f(t, x, y)$ belong to E_m, while the vector y and the vector-function $g(t, x, y)$ belong to E_{n-m}, and the vector-functions are twice continuously differentiable in some domain $D = D_1 \times E_{n-m}$ for all values $t \geq 0$, where $D_1 \subset E_m$ and $f(t, 0, 0) = g(t, 0, 0) = 0$ for $t \geq 0$. We shall denote the solution of system (1) with initial data $x(t_0) = x_0,\ y(t_0) = y_0$ by $(x(t, t_0, x_0, y_0),\ y(t, t_0, x_0, y_0))$, assuming that it exists for all $t \geq t_0$. We give the definitions of partial stability of the solution of system (1). The solution $x = 0,\ y = 0$ is called x-stable if, for any $\varepsilon > 0$ and $t_0 \geq 0$, there exists a positive number $\delta(\varepsilon, t_0)$ such that from the inequality $\|x_0\| + \|y_0\| < \delta$ for $t > t_0$ it follows that $\|x(t, t_0, x_0, y_0)\| < \varepsilon$. If $\delta(\varepsilon, t_0)$ does not depend on t_0, then x-stability is called uniform (see ([3])). Further, an x-stable solution $x = 0,\ y = 0$ is called asymptotically x-stable (a. x-st.) in the large in the domain D, if $\lim_{t\to\infty} x(t, t_0, x_0, y_0) = 0$ for all $(x_0, y_0) \in D$.

Asymptotic x-stability in the large in the domain D of the solution $x = 0,\ y = 0$ of system (1) is called asymptotic x-stability in the large in the domain D, uniformly with respect to t_0, if $\lim_{t\to\infty} x(t, t_0, x_0, y_0) = 0$ uniformly with respect to $t_0 \geq 0$ for all $(x_0, y_0) \in D$.

Asymptotic x-stability in the large in the domain D of the solution $x = 0,\ y = 0$ is called uniformly asymptotic x-stability (u.a. x-st.) in the large in the domain D, if for any closed bounded domain $\Omega \subset D$ the relation $\lim_{t\to\infty} x(t, t_0, x_0, y_0) = 0$ holds uniformly with respect to $t_0 \geq 0$ and $(x_0, y_0) \in \Omega$.

Theorem 1. In order that the solution $x = 0,\ y = 0$ of system (1) be p.a. x-stable in the large in the domain D, it is necessary and sufficient that there exist a function $V(t, x, y)$ $((x, y) \in D,\ t \geq 0)$ satisfying the following conditions:

1a) there exist continuous functions $W_i(x)$ $(i = 1, 2, 3)$ such that $W_1(x) \geq V(t, x, y) \geq W_2(x)$ for $(x, y) \in D,\ t \geq 0$, where $W_1(0) = 0$, $W_2(x) > 0$ for $x \neq 0$, and $W_2(x) \to \infty$ as $x \to \partial D_1$;

1b)

$$\frac{dV(t, x(t), y(t))}{dt} = \frac{\partial V}{\partial t} + (\operatorname{grad}_x V, f) + (\operatorname{grad}_y V, g) \leq -W_3(x) < 0$$

for $x \neq 0$, $(x, y) \in D$ and $t \geq 0$, for any solution $(x(t), y(t))$ of system (1);

1c) the function $V(t, x, y)$ has continuous partial derivatives of the second order with respect to all arguments, uniformly bounded for $t \geq 0$ and $y \in E_{n-m}$.

In the direction of sufficiency, Theorem 1 is a trivial generalization of Lyapunov' s direct theorem on asymptotic stability. In proving necessity, I. G. Malkin' s method ([2]) of constructing the Lyapunov function is used. If p.a. x-stability is not assumed, then the indicated method of constructing the function no longer applies. However, the following holds.

Theorem 2. In order that the solution $x = 0,\ y = 0$ be a. x-stable in the large in the domain D, uniformly with respect to t_0, it is necessary and sufficient that there exist a function $V(t, x(t), y(t))$ $((x, y) \in D,\ t \geq 0)$, where $x(t) = x(t, t_0, x_0, y_0)$, $y(t) = y(t, t_0, x_0, y_0)$, satisfying the following conditions:

2a) for $x' \in D_1$ and $(x, y) \in D$ there exist continuous functions $W_i(x', x, y)$ $(i = 1, 2, 3)$, with $W_i(x', x, y) > 0$ for $x' \neq 0$, such that

$$W_1(x(t), x_0, y_0) \geq V(t, x(t), y(t)) \geq W_2(x(t), x_0, y_0),$$

where $W_1(0, x_0, y_0) = 0$ and $W_2(x, x_0, y_0) \to \infty$ as $x \to \partial D_1$;

2b)

$$\frac{dV(t,x(t),y(t))}{dt} = \frac{\partial V}{\partial t} + (\operatorname{grad}_x V, f) + (\operatorname{grad}_y V, g) \leq$$
$$\leq -W_3(x, x_0, y_0) < 0 \quad (x \neq 0);$$

2c) the function $V(t,x,y)$ has continuous partial derivatives of the second order with respect to all arguments, uniformly bounded for $t \geq 0$.

3. We study in this section system (1) under the assumption that it has an a. x-stable in the large in the domain D, uniformly with respect to t_0, singular point $x = 0,\ y = 0$. Let Ω be an arbitrary closed domain from D. We shall call the number $\gamma(\Omega)$ a relaxation multiplier for the function $V(t,x,y)$ in the domain Ω, if

$$V(t,x,y) > V(t + \gamma(\Omega),\ x + \gamma(\Omega) f(t,x,y),\ y + \gamma(\Omega) g(t,x,y))$$

for $t \geq 0$ and $(x,y) \in \Omega$.

Lemma 1. For a function $V(t,x,y)$ satisfying conditions 2a)–c), in any compact domain $\Omega \subset D$ not containing the point $x = 0,\ y = 0$, there exists a relaxation multiplier.

Let now, for an arbitrary initial value $(x_0, y_0) \in \Omega$, the sequences $\{x_n\}_0^\infty,\ \{y_n\}_0^\infty$ be constructed by the Euler process

$$\begin{aligned} x_{n+1} &= x_n + \alpha_n f(t_n, x_n, y_n), \\ y_{n+1} &= y_n + \alpha_n g(t_n, x_n, y_n), \end{aligned} \tag{2}$$

where $\{\alpha_n\}_0^\infty$ is a sequence of positive numbers and

$$t_n = \sum_{k=1}^{n-1} \alpha_k.$$

Then the following holds.

Lemma 2. Let $\alpha_n \to 0$ $(n \to \infty)$. If the sequence $\{(x_n, y_n)\}_0^\infty \subset \Omega$ defined by equalities (2) and

$$\inf_{n;\ (x_0,y_0) \in D} W_1(x_n, x_0, y_0) > 0,$$

then

$$\sum_{n=0}^{\infty} \alpha_n < \infty \quad (\text{cf. } (^5)).$$

Hence it follows

Theorem 3. If $\{(x_n, y_n)\}_0^\infty \subset \Omega$, $\lim\limits_{n\to\infty} a_n = 0$, and

$$\sum_{n=0}^{\infty} a_n = \infty,$$

then

$$\lim_{n\to\infty} x_n = 0.$$

Remark 1. In the case when the function $V(t, x, y)$ satisfies conditions 1a)–c), Lemmas 1, 2 and Theorem 3 can be strengthened: instead of compactness of the domain Ω, one may require compactness of its projection Ω_x onto E_m.

4. We describe algorithms $\mathfrak{R}_1$ and $\mathfrak{R}_2$. Algorithm $\mathfrak{R}_1$ differs from algorithm $\mathfrak{R}$ (see (1)) in that the construction of the point $(x_{n+1}(j), y_{n+1}(j))$ depends on the fulfillment of the inequality

$$s_i \leq \sum_{k=1}^{n} \|x_k(j) - x_{k-1}(j)\|,$$

and the transition to the next stage is carried out when the inequality $\|x_{n+1}(j)\| > A_j$ is fulfilled, i.e., the conditions are checked with respect to the x-component.

Algorithm $\mathfrak{R}_2$ differs from $\mathfrak{R}_1$ only in the check of the condition for transition to the next stage, namely: the transition to the next stage is carried out when the inequality

$$\|(x_{n+1}(j), y_{n+1}(j))\| > A_j$$

is fulfilled.

Theorem 4. Let

$$\sup_{\substack{t\geq 0\\ y\in E_{n-m}}} \|f(t, x, y)\| < C(x) < \infty$$

and let the point $(0, 0)$ be a p.a. x-stable singular point of system (1) in the whole space $E_m \times E_{n-m}$. Then, for any initial point (x_0, y_0), algorithm $\mathfrak{R}_1$ stabilizes at some stage j_0, and the sequence it generates,

$$\{(x_n(j_0), y_n(j_0))\}_{n=0}^{\infty},$$

is such that $\lim\limits_{n\to\infty} x_n(j_0) = 0$.

The proof is based on Theorem 1 and Remark 1.

Theorem 5. Let system (1) satisfy the conditions:

a) $\sup\limits_{t\geq 0} \|f(t, x, y)\| < C(x, y) < \infty$;

b) for any (x_0, y_0) there exists $C_0(x_0, y_0)$ such that

$$\|y(t, t_0, x_0, y_0)\| < C_0(x_0, y_0),$$

and let the point $(0, 0)$ be an a. x-stable singular point, uniformly with respect to t_0, in the whole space $E_m \times E_{n-m}$. Then, for any initial point (x_0, y_0), algorithm $\mathfrak{R}_2$ stabilizes at some stage j_0, and the sequence it generates,

$$\{(x_n(j_0), y_n(j_0))\}_{n=0}^{\infty},$$

is such that $\lim\limits_{n\to\infty} x_n(j_0) = 0$ and $\{y_n(j_0)\}_{n=0}^{\infty}$ is contained in a bounded domain.

The proof uses Theorems 2, 3.

Corollary. Algorithm $\mathfrak{R}_2$ is applicable to finding the value x^0 corresponding to the saddle point (x^0, y^0) of a functional $\varphi(x, y)$, strictly concave in the variables x and convex in the variables y, since the corresponding system for $\varphi(x, y)$

$$dx/dt = \operatorname{grad}_x \varphi(x, y), \qquad dy/dt = -\operatorname{grad}_y \varphi(x, y)$$

satisfies the conditions of Theorem 5 (see (6)).

In conclusion, the authors gratefully note the constant attention of I. M. Glazman during the writing of this note.

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Note: Figure translations are in progress. See original paper for figures.