ON THE STABILITY OF MOTION ON A FINITE TIME INTERVAL

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Abstract

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MECHANICS

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ON THE STABILITY OF MOTION ON A FINITE TIME INTERVAL

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1. We consider systems whose perturbed motion is described by an equation of the form

$$dx/dt = U(t)x + H(t, x), \tag{1}$$

where x is a column matrix composed of the deviations x_i $(i = 1, \ldots, n)$ of the system, and U and H are matrices of type $n \times n$ and $n \times 1$, respectively, possessing on the set $S = [t_0 \leq t \leq t_0 + T,\ \|x\| \leq a]$ the properties: a) U is differentiable l times $(l = 1, 2, \ldots)$; b) H is continuous in t and x;

$$\lim_{x \to 0} \frac{H(t, x)}{\|x\|} = 0. \tag{2}$$

For such systems there is given below a family of necessary and sufficient conditions for stability and instability of the unperturbed motion (the trivial solution of equation (1)) on a finite time interval, proceeding from the following definition of stability.

Definition. If the equations of perturbed motion are such that, for sufficiently small $\rho > 0$, any solution $x(t)$ of the equations whose initial value $x_0 = x(t_0)$ satisfies the condition

$$(G(t_0)x_0,\ G(t_0)x_0) \leq \rho, \tag{3}$$

on some finite interval $[t_0,\ t_0 + \Delta t]$ satisfies the condition

$$(G(t)x,\ G(t)x) \leq \rho, \tag{4}$$

where G is a given bounded matrix, then the unperturbed motion with respect to the domain (4) is stable on the interval $[t_0,\ t_0 + \Delta t]$; otherwise it is unstable, i.e. $\Delta t = 0$.

Unlike the definition of stability given by G. V. Kamenkov ([1]), here, as in works ([2,3]) (but with different restrictions), a variable domain of limiting deviations x_i $(i = 1, \dots, n)$ is introduced.

2. Instead of (1) let us consider the more general equation

$$dx/dt = U(\tau)x + H(t, x), \tag{5}$$

where $\tau = \varepsilon t$ is the so-called "slow time," and ε is a parameter. When $\varepsilon = 1$, equations (1) and (5) coincide.

We shall assume that $U(\varepsilon t)$ on the set S is still differentiable l times.

By the transformation

$$x = K^{(m)}(\tau, \varepsilon)y \tag{6}$$

we bring equation (5) to the form

$$dy/dt = \Lambda^{(m)}(\tau, \varepsilon)y - K^{(m)-1}(\tau, \varepsilon)N^{(m)}(\tau, \varepsilon) + K^{(m)-1}(\tau, \varepsilon)H(t, K^{(m)}y), \tag{7}$$

where

$$N^{(m)}(\tau, \varepsilon) = \varepsilon \frac{dK^{(m)}(\tau, \varepsilon)}{d\tau} - U(\tau)K^{(m)}(\tau, \varepsilon) + K^{(m)}(\tau, \varepsilon)\Lambda^{(m)}(\tau, \varepsilon). \tag{8}$$

Using the algorithm indicated in (4), one can construct such a transformation (6) that $\Lambda^{[m]}$ will have a diagonal or, at least, a quasidiagonal structure, while $N^{(m)}$, as $\varepsilon \to 0$, will be a matrix of order not lower than ε^{m+1}.

We shall restrict ourselves to the case when $U(\varepsilon t)$ has only simple eigenvalues on $[t_0, T]$. Then $K^{(m)}$ and $\Lambda^{(m)}$ can be represented as follows:

$$K^{(m)}(\tau, \varepsilon) = \sum_{k=0}^{m} \varepsilon^k K^{[k]}(\tau), \qquad \Lambda^{(m)}(\tau, \varepsilon) = \sum_{k=0}^{m} \varepsilon^k \Lambda^{[k]}(\tau), \tag{9}$$

where

$$K^{[k]} = (K_1^{[k]} \dots K_n^{[k]}), \qquad \Lambda^{[k]} = \begin{pmatrix} \lambda_1^{[k]} & 0 & \\ & \ddots & \\ 0 & & \lambda_n^{[k]} \end{pmatrix}.$$

By $\lambda_1, \dots, \lambda_n$ we denote the eigenvalues of the matrix U, and by $K_1, \dots, K_n$ the corresponding normalized eigenvectors of it. Then the column matrices $K_\sigma^{[k]}$ and the scalar functions $\lambda_\sigma^{[k]}$ are determined by the recurrence relations

$$K_\sigma^{[0]} = K_\sigma, \qquad \lambda_\sigma^{[0]} = \lambda_\sigma, \qquad K_\sigma^{[k]} = P_\sigma D_\sigma^{[k-1]} + K_\sigma q_\sigma^{[k]},$$

$$\lambda_\sigma^{[k]} = -M_\sigma D_\sigma^{[k-1]} \qquad (k = 1, \dots, m;\ \sigma = 1, \dots, n), \tag{10}$$

where

$$P_\sigma = \sum_{s \neq \sigma} \frac{K_s M_s}{\lambda_s - \lambda_\sigma}, \qquad D_\sigma^{[k-1]} = \sum_{\alpha=1}^{k-1} K_\sigma^{[k-\alpha]} \lambda_\sigma^{[\alpha]} + \frac{dK_\sigma^{[k-1]}}{dt}, \tag{11}$$

M_s $(s = 1, \dots, n)$ are the rows of the matrix

$$M = \begin{pmatrix} M_1 \\ \vdots \\ M_n \end{pmatrix} = K^{-1} = (K_1 \dots K_n)^{-1},$$

and $q_\sigma^{[k]}$ are arbitrary functions of τ, differentiable $m - k + 1$ times. With this construction of $K^{(m)}$ and $\Lambda^{(m)}$,

$$N^{(m)} = \varepsilon^{m+1} \sum_{\nu=1}^{m} \sum_{\alpha=\nu}^{m} \varepsilon^{\nu-1} K^{[m-\alpha+\nu]} \Lambda^{[\alpha]} \equiv \varepsilon^{m+1} N_0^{(m)}. \tag{12}$$

The freedom available in the choice of $q_\sigma^{[k]}$ can be used for "normalizing" the columns of the matrix $K^{(m)}$. If, for example, one sets

$$q_\sigma^{[k]} = -K_\sigma^* P_\sigma D_\sigma^{[k-1]} - \frac{1}{2} \sum_{\alpha=1}^{k-1} K_\sigma^{[k-\alpha]*} K_\sigma^{[\alpha]} \tag{13}$$

$$(k = 1, \dots, m;\ \sigma = 1, \dots, n),$$

then the Euclidean norm of the columns of the matrix $K^{(m)}$, up to quantities of order ε^{m+1}, will be equal to one. In what follows, unless otherwise specified, we shall assume that the columns of the matrix $K^{(m)}$ possess the indicated property.

3. We define the domain of limiting deviations by the relation

$$V(\tau, \varepsilon, x) \equiv ((K^{(m)})^{-1}(\tau, \varepsilon)x,\ K^{(m)-1}(\tau, \varepsilon)x) \leq \rho. \tag{14}$$

Geometrically, the domain (14) is an n-dimensional ellipsoid bounded by the surface $V(\tau, \varepsilon, x) = \rho$. Each of the $2n$ rays $x = \pm K(\tau, \varepsilon)s$ ($\sigma = 1, \dots, n;\ 0 < s \leq \infty$) intersects this surface once, at the value of the parameter $s = \sqrt{\rho}$. Up to quantities of order ε^{m+1}, the points of intersection are at a constant distance $\sqrt{\rho}$ from the origin ($x = 0$).

We shall determine the conditions of stability and instability of the trivial solution of equation (5) with respect to the domain (14).

From (7) we find

$$\frac{d\|y\|}{dt} = \sum_{\sigma=1}^{n} \operatorname{Re} \lambda_\sigma^{(m)} \frac{|y_\sigma|^2}{\|y\|} + \varepsilon^{m+1} \frac{y^* \mathcal{P}^{(m)} y}{\|y\|} + \frac{1}{2\|y\|} \operatorname{Re}\{y^* K^{(m)-1} H\}, \tag{15}$$

where

$$\mathcal{P}^{(m)} = -\frac{1}{2} \left[K^{(m)-1} N_0^{(m)} + \left(K^{(m)-1} N_0^{(m)} \right)^* \right],$$

and y_σ ($\sigma = 1, \dots, n$) are the elements of the column matrix y.

In accordance with expressions (6) and (15), the derivative of the positive definite function V with respect to t, computed by virtue of equation (5), is equal to

$$\frac{1}{2} \frac{dV}{dt} \sum_{\sigma=1}^{n} \operatorname{Re} \lambda_\sigma^{(m)} |y_\sigma|^2 + \varepsilon^{m+1} y^* \mathcal{P}^{(m)} y + \frac{1}{2} \operatorname{Re}\{y^* K^{(m)-1} H\}. \tag{16}$$

Let

$$\mu^{(m)}(\tau, \varepsilon) = \max_\sigma \left(\operatorname{Re} \lambda_\sigma^{(m)}(\tau, \varepsilon) \right);$$

$v_{\min}^{(m)}(\tau, \varepsilon)$, $v_{\max}^{(m)}(\tau, \varepsilon)$ are, respectively, the minimum and maximum eigenvalues of the Hermitian matrix $\mathcal{P}^{(m)}$.

Theorem 1. *If*

$$\mu^{(m)}(\tau_0, \varepsilon) + \varepsilon^{m+1} v_{\max}^{(m)}(\tau_0, \varepsilon) < 0 \qquad (\tau_0 = \varepsilon t_0), \tag{17}$$

then the trivial solution of equation (5) is stable on the finite interval $[t_0,\ t_0 + \Delta t]$.

Proof. From condition (2) and the boundedness of $K^{(m)}$ it follows that, uniformly in t on the segment $[t_0, T]$,

$$\lim_{y \to 0} \frac{H(t, K^{(m)} y)}{\|y\|} = 0. \tag{18}$$

Taking this into account, from (16) we obtain

$$\frac{1}{2}\frac{dV}{dt} \leq \left(\mu^{(m)}(\tau,\varepsilon) + \varepsilon^{m+1} v_{\max}^{(m)}(\tau,\varepsilon)\right) \|y\|^2 + o(\|y\|^2).$$

It is therefore clear that, if inequality (17) holds, then for sufficiently small $\|y\|$ at the point $t = t_0$, and by continuity also within some finite interval

$$[t_0,\, t_0 + \Delta t] \subset [t_0, T],$$

$$dV/dt < 0,$$

which proves the theorem.

Theorem 2. *If*

$$\mu^{(m)}(\tau_0,\varepsilon) + \varepsilon^{m+1} v_{\min}^{(m)}(\tau_0,\varepsilon) > 0, \tag{19}$$

then the trivial solution of equation (5) is not stable on the finite interval $[t_0,\, t_0 + \Delta t]$*, i.e.* $\Delta t = 0$*.*

Proof. Integrating (15), we obtain

$$\|y(t,\varepsilon)\| = \|y(t_0,\varepsilon)\| \exp\left\{\int_{t_0}^{t}\left[\sum_{\sigma=1}^{n} \operatorname{Re}\lambda_\sigma^{(m)} \frac{|y_\sigma|^2}{\|y\|^2} + \varepsilon^{m+1}\frac{y^*\mathcal{P}^{(m)}y}{\|y\|^2} + O(\|y\|)\right] dt\right\}. \tag{20}$$

If

$$\varphi(\tau,\varepsilon,y) \equiv \sum_{\sigma=1}^{n} \operatorname{Re}\lambda_\sigma^{(m)} \frac{|y_\sigma|^2}{\|y\|^2} + \varepsilon^{m+1}\frac{y^*\mathcal{P}^{(m)}y}{\|y\|^2} \neq 0,$$

then, for sufficiently small $\|y\|$, the sign of the integrand coincides with the sign of the function φ.

Suppose that $\mu^{(m)}(\tau_0,\varepsilon) = \operatorname{Re}\lambda_s^{(m)}(\tau_0,\varepsilon)$ and that $\bar{x} = K^{(m)}\bar{y}$ is a particular solution of equation (5), determined by the initial conditions $y_s(t_0,\varepsilon) = \sqrt{\rho}$, $y_\sigma(t_0,\varepsilon) = 0$ $(\sigma \neq s)$. For this solution,

$$\varphi(\tau_0,\varepsilon,\bar{y}(t_0,\varepsilon)) \geq \mu^{(m)}(\tau_0,\varepsilon) + \varepsilon^{m+1}\nu_{\min}^{(m)}(\tau_0,\varepsilon) > 0.$$

Therefore, at the point t_0, for sufficiently small ρ, the integrand in equality (20) is positive. By continuity it is positive also in some neighborhood of the point t_0. Hence, in this neighborhood,

$$dV(\tau, \varepsilon, \bar{x})/dt = 2\|\bar{y}\|\, d\|\bar{y}\|/dt > 0,$$

and therefore, along the indicated solution, condition (4) is not satisfied.

Theorem 3. *If*

$$\mu^{(m)}(\tau_0, \varepsilon) + \varepsilon^{m+1}\nu_{\min}^{(m)}(\tau_0, \varepsilon) \le 0 \le \mu^{(m)}(\tau_0, \varepsilon) + \varepsilon^{m+1}\nu_{\max}^{(m)}(\tau_0, \varepsilon), \tag{21}$$

then the trivial solution of equation (5) may fail to possess stability on the finite interval $[t_0, t_0 + \Delta t]$.

Proof. The inequalities (21) allow the existence of a particular solution $\bar{x} = K^{(m)}\bar{y}$ such that

$$\varphi(\tau_0, \varepsilon, \bar{y}(t_0, \varepsilon)) = 0, \qquad \|\bar{y}(t_0, \varepsilon)\| = \sqrt{\rho}.$$

For this solution the sign of the integrand in equality (20) is determined by the sign of $O(\|y\|)$, so that, depending on the properties of the nonlinear terms at $t = t_0$, and by continuity also within the limits of some neighborhood of the point t_0, the integrand may be positive. Then, in this neighborhood,

$$dV(\tau, \varepsilon, \bar{x})/dt > 0,$$

and, consequently, condition (4) will not be satisfied.

4. Applying the results of item 3 to equation (1), we shall have:

$$\begin{aligned} \mu^{(m)}(t_0) + \nu_{\max}^{(m)}(t_0) < 0 \quad & \text{—sufficient condition for stability,} \\ \mu^{(m)}(t_0) + \nu_{\min}^{(m)}(t_0) \le 0 \quad & \text{—necessary condition for stability,} \\ \mu^{(m)}(t_0) + \nu_{\min}^{(m)}(t_0) > 0 \quad & \text{—sufficient condition for instability,} \\ \mu^{(m)}(t_0) + \nu_{\max}^{(m)}(t_0) \ge 0 \quad & \text{—necessary condition for instability.} \end{aligned} \tag{22}$$

Here

$$\mu^{(m)}(t) = \mu^{(m)}(\tau, \varepsilon)\big|_{\varepsilon=1}, \qquad \nu_{\min,\max}^{(m)}(t) = \nu_{\min,\max}^{(m)}(\tau, \varepsilon)\big|_{\varepsilon=1}.$$

The inequalities (22) constitute a whole family of necessary and sufficient conditions corresponding to the numbers $m = 0, 1, 2, \dots, l-1$.

The criteria (22), for a given m, do not solve the problem in those cases when $\mu^{(m)}$ lies within the strip

$$-\nu_{\min}^{(m)} \ge \mu^{(m)} \ge -\nu_{\max}^{(m)}. \tag{23}$$

With increasing m (at least up to a certain value) one may expect a substantial reduction in the width of the "strip of indeterminacy," especially when the coefficients of the equations of the first approximation are slowly varying functions of t.

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Note: Figure translations are in progress. See original paper for figures.