AN AGEOSTROPHIC TWO-LAYER MODEL OF BAROCLINIC INSTABILITY IN THE ATMOSPHERE AND OCEAN

GEOPHYSICS

1968

SovietRxiv

Abstract

Full Text

UDC 532.32

GEOPHYSICS

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AN AGEOSTROPHIC TWO-LAYER MODEL OF BAROCLINIC INSTABILITY IN THE ATMOSPHERE AND OCEAN

(Presented by Academician A. A. Dorodnitsyn, 22 IV 1967)

N. Phillips ([1]) showed that a two-layer quasi-geostrophic model of baroclinic instability gives a good quantitative approximation to the results of quasi-geostrophic theories ([2], [3]) and others, which take into account the continuous variation of the velocity and density of the basic current with height. This is connected with the fact that the processes of release of the potential energy of horizontal stratification, which cause baroclinic instability, do not depend essentially on the detailed structure of the velocity profile of the basic current. In the two-layer models of Solberg ([4]) and Høiland ([5]), for the sake of mathematical simplification it was assumed that the rigid walls bounding the upper and lower fluid layers are parallel to the interface. Such boundary conditions automatically filter out gradient-vortex waves associated with changes in layer thickness and responsible for baroclinic instability. The instability of internal gravity waves found in these models is determined by the kinetic energy of the basic motion and is a type of Helmholtz instability, depending decisively on the form of the velocity profile (a velocity jump at the interface). Since dissipative factors smooth out discontinuous changes of the hydrodynamic elements at the discontinuity surface, Helmholtz instability does not play an important role in the dynamics of large-scale motions, and therefore the results of the Norwegian school did not lead to the creation of a physical theory of cyclogenesis. The theory of N. E. Kochin ([6]) takes into account changes in layer thickness and gradient-vortex waves, but nevertheless its results are in contradiction with ([1–3]) and others. In quasi-geostrophic models the instability increases with increasing vertical shear of the velocity of the basic current, whereas in Kochin' s theory instability arises when the velocity shear decreases (when the Richardson number increases). However, this contradiction is only apparent, and below a model is proposed that unites and generalizes the results of Phillips and Kochin. It turns out that Kochin' s result is connected partly with the peculiarities of the geometry of the model (the frontal surface intersects the bottom and the free surface, so that the width of the front is not an independent parameter), and chiefly with the fact that Kochin' s neutral curve corresponds to a substantially ageostrophic regime and bounds the region of instability from above (with respect to the velocity shear).

The supposition of the existence of an upper boundary of instability associated with the influence of ageostrophy was also made by Arnason ([7]), who investigated a continuous model, but this supposition is mainly qualitative in character (the power series used in ([7]) to calculate the neutral curve cease to converge in the region of ageostrophy). A two-layer ageostrophic model with the corresponding mathematical simplifications makes it possible not only to obtain neutral curves, but also, by simple means, to investigate the internal structure of the region of instability.

Consider a zonal two-layer current of width l, in which the inclination of the undisturbed interface is determined by the Margules formula

$$\operatorname{tg}\delta = f(U_2 - U_1)g', \qquad g' = g(\rho_2 - \rho_1/\rho).$$

The indices $1, 2$ here and below refer respectively to the lower and upper layers; ρ is the mean value of the potential density; a coordinate system on the β-plane is used ($f = \beta y + f_0$); the z-axis is directed upward. The linearized equations of the disturbed motion have the form

$$\frac{d_j u_j'}{dt} - f v_j' = -\frac{1}{\rho}\frac{\partial p_j'}{\partial x}; \qquad \frac{d_j v_j'}{dt} + f u_j' = -\frac{1}{\rho}\frac{\partial p_j'}{\partial y}; \tag{1}$$

$$d_j\zeta'/dt + v_j' \operatorname{tg}\delta + \varepsilon_j D_j(\partial u_j'/\partial x + \partial v_j'/\partial y) = 0; \tag{2}$$

$$\rho g'\zeta' = p_1' - p_2'; \qquad (d_j/dt = \partial/\partial t + U_j\partial/\partial x; \quad \varepsilon_1 = -\varepsilon_2 = 1; \quad j = 1, 2). \tag{3}$$

Here ζ' is the perturbation of the interface; D_j are the layer depths; equation (3) is a consequence of the conditions of hydrostatics and continuity of the total pressure at the interface. In deriving the continuity equation (2) it is assumed that the free surface of the upper layer is immobile (this assumption filters out barotropic long waves, which are of no interest for the problem). From (1) we compose the vorticity equation and replace the value of the horizontal divergence using (2):

$$\frac{d_j\Omega_j'}{dt} + \beta v_j' = \varepsilon_j \frac{f}{D_j}\left(\frac{d_j\zeta'}{dt} + v_j'\operatorname{tg}\delta\right), \qquad \Omega_j' = \frac{\partial v_j'}{\partial x} - \frac{\partial u_j'}{\partial y}. \tag{4}$$

If the dependent variables are represented in the form $\psi_j' = \psi_j(y) \times \exp[i(kx - \omega t)]$, then from (1) we obtain

$$u_j = \frac{(kU_j - \omega)kp_j - f\,dp_j/dy}{f^2 - (kU_j - \omega)^2}; \qquad v_j = i\frac{fkp_j - (kU_j - \omega)dp_j/dy}{f^2 - (kU_j - \omega)^2}. \tag{5}$$

Passing to a coordinate system moving along the x-axis with velocity

$$\overline{U} = (U_1 + U_2)/2,$$

and introducing the notation

$$U = (U_2 - U_1)/2; \qquad \mathcal{F} = \frac{k^2U^2}{f^2}, \qquad c = \frac{\bar{\omega}}{k}\Big/U, \qquad \bar{\omega} = \omega - k\overline{U},$$

from (4), (5), and (3) (taking into account that $\partial D_1/\partial y = -\partial D_2/\partial y = \operatorname{tg}\delta$), we obtain a system of equations for p_1, p_2:

$$\frac{d^2p_1}{dy^2} + \frac{\operatorname{tg}\delta}{D_1}\frac{dp_1}{dy} - \left[k^2 + \frac{\beta}{U(1+c)} - \frac{\operatorname{tg}\delta}{D_1}\frac{f}{U(1+c)}\right]p_1 = \frac{f^2[1 - \mathcal{F}(1+c)^2]}{g'D_1}(p_1 - p_2)$$

$$\frac{d^2p_2}{dy^2} - \frac{\operatorname{tg}\delta}{D_2}\frac{dp_2}{dy} - \left[k^2 - \frac{\beta}{U(1-c)} - \frac{\operatorname{tg}\delta}{D_2}\frac{f}{U(1-c)}\right]p_2 = \frac{f^2[1 - \mathcal{F}(1-c)^2]}{g'D_2}(p_2 - p_1). \tag{6}$$

If the changes in the depths of the layers over the width of the current are small in comparison with the depths themselves, then the second terms in the left-hand sides of (6) are negligibly small and, moreover, $D_j = D_j(y)$ may be replaced by constant mean values. In order to filter out internal gravitational waves but retain the influence of ageostrophy (associated with the parameter $\mathcal{F}$) on gradient-vorticity waves, one should put $c = 0$ in the right-hand sides of (6). The boundary conditions (in the quasi-geostrophic approximation) have the form $p_j(0) = p_j(l) = 0$ in the case of rigid walls bounding the current, or $dp_j/dy|_{y=0} = dp_j/dy|_{y=l} = 0$ in the case of free boundaries, beyond which the fluid is at rest. Taking into account the approximations indicated above, the eigenfunctions will be respectively

$$p_j = A_j \sin\lambda_n y \qquad \text{or} \qquad p_j = A_j \cos\lambda_n y, \qquad \text{where } \lambda_n = n\pi/l.$$

The quantities A_j must satisfy a system of linear homogeneous equations. Equating to zero the determinant of this system,

$$\begin{vmatrix} \left[\lambda_n^2 + k^2 - \dfrac{2f^2/g'D_1 - \beta/U}{1+c} + \left(\dfrac{f^2}{g'D_1} - k^2\dfrac{U^2}{g'D_1}\right)\right] & -\left(\dfrac{f^2}{g'D_1} - k^2\dfrac{U^2}{g'D_1}\right) \\ -\left(\dfrac{f^2}{g'D_2} - k^2\dfrac{U^2}{g'D_2}\right) & \left[\lambda_n^2 + k^2 - \dfrac{2f^2/g'D_2 + \beta/U}{1-c} + \left(\dfrac{f^2}{g'D_2} - k^2\dfrac{U^2}{g'D_2}\right)\right] \end{vmatrix}$$

we obtain a quadratic equation for determining c. Complex values of c correspond to instability. In the particular case of a broad current ($lk_0 \gg \pi$, where $k_0 = \sqrt{f^2/g'D/2}$) and equal layer thicknesses $D_1 = D_2 = D$, the expression for the complex velocity in a fixed coordinate system has the form

$$\frac{\omega_r + i\omega_i}{k} = \overline{U} - \frac{\beta[(1-\alpha F) + 2\alpha]}{2k^2[(1-\alpha F)+\alpha]} \pm \frac{1}{2\alpha[(1-\alpha F)+\alpha]} \times$$

$$\times \sqrt{(\beta^2/k_0^4)(1-\alpha F)^2 + (U_2 - U_1)^2\alpha^2(\alpha - 1)[\alpha(1-F)^2 + (1-F)]}. \quad (7)$$

Here $\alpha = k^2/k_0^2$; $F = \mathrm{Ri}^{-1} = \dfrac{U^2}{g'D/2}$ is the reciprocal of the Richardson number in the two-layer model. In the case $F = 0$ we obtain Phillips' result[1]. Figure 1 shows the region of instability, calculated from formula (7), for middle latitudes ($\varphi \sim 40^\circ$, i.e. $f \sim 10^{-4}$ sec^{-1}). Along the abscissa axis are plotted wavelengths in units of $L_0 = 2\pi/k_0$, and along the ordinate axis—the values of the velocity shear $(U_2 - U_1)$ in units of $\sqrt{g'H}$, where $H = 2D$ is the effective thickness of the atmosphere or the total depth of the ocean. Diagrams for the more complicated cases $D_1 \neq D_2$ and a current of finite width are not given here. The isolines inside the region of instability correspond to dimensionless values of $\omega_i/2f$, increased by a factor of 10^2. Whereas in the lower part of the figure allowance for ageostrophy gives only small quantitative corrections to the quasigeostrophic models, for small Richardson numbers the ageostrophic effects stabilize the motion, and with a further increase in the velocity shear the motion again becomes unstable. Thus, there exists not only a "dominant wavelength," but also a "dominant velocity shear," corresponding to the most rapid growth of disturbances. This result has far-reaching consequences and makes it possible to understand certain facts that seemed inexplicable within the narrow framework of quasigeostrophic theory. Thus, for example, the absence of meanders and large-scale vortices in the Gulf Stream region from Florida to Cape Hatteras becomes understandable, since there the ratio $(U_2 - U_1)/\sqrt{g'H}$ is of order unity. For typical atmospheric parameters $\sqrt{g'H} \sim$ 40–50 m/sec, so that the influence of ageostrophy on the behavior of disturbances can be substantial only for very large values of velocity shear, characteristic in particular of jet streams.

Fig. 1

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Received
14 IV 1967

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Note: Figure translations are in progress. See original paper for figures.