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Abstract

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MATHEMATICS

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ON THE ABSENCE OF AN UNCONDITIONAL BASIS

IN THE QUOTIENT SPACES $L_1^{'}/H_0^1$ AND $L^{'}/H_0^{'}$

(Presented by Academician A. N. Kolmogorov on 19 V 1967)

Up to now Banach' s problem has not been solved: does a basis exist in an arbitrary separable Banach space? As for unconditional bases, S. Karlin ([2]) was the first to prove that in the space $C[0,1]$ of continuous functions on the segment $[0,1]$ there is no unconditional basis. The absence of an unconditional basis in the space $L_1[0,1]$ of all real absolutely summable functions on the segment $[0,1]$ was shown by A. Pełczyński ([3]); he also posed the following question: does an unconditional basis exist in the quotient space L_1/H_0^1? The present paper gives a negative answer to this question. The absence of an unconditional basis in the quotient space L'/H_0' is also proved. In addition, it is shown that each of the indicated quotient spaces is not isomorphic to any subspace of a separable Banach space X^* conjugate to a Banach space X, nor to any subspace of a Banach space with an unconditional basis.

We shall use the following notation: H^1 is the Hardy space of functions $f(z)$, analytic in the unit disk and such that

$$\sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})| \, d\theta = \|f\| < \infty, \qquad z = re^{i\theta},$$

$$H_0^1 = \{f \in H^1 : \; f(0) = 0\};$$

H_0' is the space of functions analytic in the unit disk, vanishing at the origin, and such that

$$\int_0^1 \int_0^{2\pi} |f(re^{i\theta})| \, r \, dr \, d\theta = \|f\| < \infty;$$

H_∞ is the B-space of analytic bounded functions in the unit disk with norm

$$\|f\|_\infty = \operatorname*{ess\,sup}_{|z|=1} |f(z)|;$$

L' is the B-space of summable functions in $|z| < 1$.

It is known ([5], p. 195) that $(L_1/H_0^1)^* = H_\infty$. The space conjugate to L'/H_0' is, as is known ([1]), the annihilator of the space H_0', i.e. the set of those essentially bounded functions in the unit disk for which

$$\iint_{|z|<1} z^n f(z)\, d\sigma = 0, \qquad z = x + iy, \qquad d\sigma = dx\, dy, \qquad n = 1, 2, \ldots$$

Definition. A basis in a B-space is called **unconditional** if every system obtained by permuting its elements is also a basis.

Theorem 1. *In the quotient space L_1/H_0^1 there is no unconditional basis.*

The proof rests on a number of lemmas.

Lemma 1. Let E be a separable subspace of H_∞. Then there exists a perfect set $T \subset [0, 2\pi]$ of positive measure such that every function $\varphi \in E$ is equivalent to a function whose restriction to T is continuous.

Lemma 2. Let Z be a B-space satisfying one of the following conditions: a) Z is separable and is the conjugate of some B-space ($Z = X^*$); b) Z contains an unconditional basis.

Then for every subspace $Y \subset Z$ there exists a separable subspace $E_Y \subset Y^*$ such that for every $y_0^* \in Y^*$ there exists a sequence $\{y_n^*\} \subset E_Y$ such that:

$$\text{(I)} \qquad y_0^*(y) = \lim_{n\to\infty} y_n^*(y) \quad \text{for every } y \in Y;$$

$$\text{(II)} \qquad \text{for every } y^{**} \in Y^{**} \text{ there exists } \lim_{n\to\infty} y^{**}(y_n^*).$$

Lemma 3. Let $T \subset [0, 2\pi]$ be a perfect set of positive measure. Then there exists a function $\varphi_0(z) \in H_\infty$ whose restriction of angular boundary values to the set T is not equivalent to any function of the first Baire class.

Lemmas 1 and 2 were proved by Pelczyński ([3]); the proof of Lemma 3 will be given below.

For a measurable function put

$$\Omega(\varphi, \Delta) = \sup_{t',\, t'' \in \Delta} |\varphi(t') - \varphi(t'')|, \qquad \operatorname{ess}\Omega(\varphi, \Delta) = \inf_{\psi\sim\varphi} \Omega(\psi, \Delta),$$

$$\operatorname{ess}\Omega(\varphi, t) = \lim_{\Delta_k \to t} \operatorname{ess}\Omega(\varphi, \Delta_k).$$

Let $\{t_n = e^{i\theta_n}\}$ be a countable everywhere dense set in T. Consider the following sets of functions:

$$F = \{\varphi \in H_\infty : \|\varphi\|_\infty \le 1\},$$

$$F_n = \{\varphi \in F : \operatorname{ess}\Omega(\varphi, t_n) \le \alpha < 1/10\}, \qquad n = 0, 1, \dots,$$

$$F^i_{n,k} = \{\varphi \in F : \operatorname{ess}\Omega(\varphi, \Delta_{n,k}) \le \alpha + 1/i\}, \qquad i = 5, 6, \dots,\ t_n \in \Delta_{n,k}.$$

Lemma 3^1**.** The set F with the metric

$$\rho(\varphi_1, \varphi_2) = \frac{1}{2\pi} \int_0^{2\pi} \frac{|\varphi_1(e^{i\theta}) - \varphi_2(e^{i\theta})|}{1 + |\varphi_1(e^{i\theta}) - \varphi_2(e^{i\theta})|}\, d\theta$$

is a complete metric space.

Lemma 3^2**.** The set $F^i_{n,k}$ is closed.

Lemma 3^3**.** The set $F^i_{n,k}$ is nowhere dense in F.

In order to prove this, it is enough to verify that for every $\varepsilon > 0$ and every function $f \in F^i_{n,k}$ one can find a function $g \in F$, lying in the ε-neighborhood of f, such that g does not belong to $F^i_{n,k}$. Thus, Lemma 3^3 will be proved if the following is true.

Assertion 1. Let Δ be some interval of $|z| = 1$, and let $\beta > 0$ be a fixed number. Then for every $\varepsilon > 0$ and every function $f(z) \in F$ satisfying the condition $\operatorname{ess}\Omega(f, \Delta) < \beta < 1/3$, there exists a function $g(z)$ such that

$$g(z) \in F, \qquad \rho(f, g) < \varepsilon, \qquad \operatorname{ess}\Omega(g, \Delta) > \beta.$$

Proof. Suppose that $\|f\|_\infty = 1$ (the case $\|f\|_\infty < 1$ introduces nothing new). Consider the sequence of functions $\{f_n(z) = f(r_n z)\}$, $r_n < 1$. Each of these functions is analytic in $|z| \le 1$.

As $r_n \uparrow 1$, the sequence $\{f_n(z)\}$ converges to the function $f(z)$ in the norm H^1 ([4], p. 89), and hence also in measure. Moreover, $\max_{|z|=r} |f(z)|$, strictly increasing, tends to one. Therefore, for the given $\varepsilon > 0$ (we shall assume $\varepsilon < 1/2$) one can indicate a number N such that the inequalities

$$\rho(f_N, f) < \varepsilon/4, \qquad 1 - l_N = \delta < \varepsilon, \qquad \text{where } l_N = \|f_N\|_\infty < 1.$$

Let us now make the following construction. With center at some interior point $z_1 \in \Delta$, take an arc σ so small that $0 < m\sigma < \delta/4$, and such that for any points $z', z'' \in \sigma$ (including also the endpoints of the interval σ, which we denote by the points z_2, z_3) the condition

$$|f_N(z') - f_N(z'')| < \delta/8$$

is satisfied.

Let $w_1' = f_N(z_1)$. Join the point w_1' to the origin; draw two circles $|w| = \delta/2$, $|w| = \beta + 3\delta/4$, and two rays symmetric with respect to Ow_1' at sufficiently small angles to it. The domain bounded by these rays, by the larger arc of the circle $|w| = \delta/2$, and by the smaller arc of the circle $|w| = \beta + 3\delta/4$, will be denoted by G, its boundary by Γ, the points of intersection of the rays with the circle $|w| = \delta/2$ by w_2, w_3, and the point of intersection of the circle $|w| = \beta + 3\delta/4$ with the ray Ow_1' by w_1.

Let $\mu(z)$ be a function effecting a conformal mapping of $|z| \leq 1$ onto the domain G with the additional conditions: the points z_i pass into the points w_i $(i = 1, 2, 3)$. Such a function exists (by Riemann' s theorem), is analytic in $|z| < 1$, and continuous in $|z| \leq 1$. It is then shown that the function $g(z) = f_N(z) - \mu(z)$ is the required one.

Completion of the proof of Lemma 3. The inclusion

$$F_n \subset \sum_{k=1,\, i=5}^{\infty} F_{n,k}^i$$

is valid. Since $F_{n,k}^i$ is nowhere dense in F, it follows that F_n is a set of the first category;

$$\sum_{n=1}^{\infty} F_n$$

is also a set of the first category, and consequently the set

$$F_0 = F - \sum_{n=1}^{\infty} F_n$$

is nonempty. Suppose that $\varphi_0 \in F_0$. Then for it $\operatorname{ess}\Omega(\varphi_0, t_n) > \alpha,\ n = 1, 2, ...$, and since $\{t_n\}$ is everywhere dense in T, no point of the set T is a point of continuity of the function φ_0 and of its equivalents. The lemma is proved.

The proof of Theorem 1 is now carried out according to a scheme analogous to Pelczyński' s scheme ([3]).

Theorem 2. *In the quotient space L'/H_0' there is no unconditional basis.*

The general scheme of the proof of Theorem 2 is analogous to the scheme of the proof of Theorem 1. The difference will occur only in the proof of the main assertion.

Assertion 2. *Let Δ be some plane set of positive measure in $|z| < 1$; let $\beta > 0$ be a fixed number. Then for every $\varepsilon > 0$ and every function $f \in F$ satisfying the condition*

$$\operatorname{ess}\Omega(f, \Delta) \leq \beta < 1/10,$$

there exists a function $g(z)$ such that

$$g(z) \in F, \qquad \rho(f, g) < \varepsilon, \qquad \operatorname{ess}\Omega(g, \Delta) > \beta.$$

Proof. Among the points of the set Δ there exists at least one point z_1, in an arbitrarily small neighborhood of which there is a set of positive measure from Δ. For the number $\varepsilon/2$ and the function $f(z)$ construct a piecewise-constant function $f_0(z) \in F$ close to $f(z)$. Take, besides the point z_1, also some point z_2, and draw a circle through them.

Moreover, choose the point z_2 so that the area of the disk K (bounded by the circle passing through z_1, z_2) is less than $\varepsilon/2$.

Let $w_1 = f_0(z_1)$. Join the point w_1 to the origin and draw from the origin two rays symmetric with respect to Ow_1, making sufficiently small angles with it. We consider three cases: the 1st case $|w_1| \geq 1/2$; the 2nd case $|w_1| < 1/2,\ |w_1| \neq 0$; the 3rd case $|w_1| = 0$.

In the 1st case denote by D the domain bounded by the two indicated rays and by the smaller arcs of the circles $|w| = |w_1|$ and $|w| = |w_1| - 1/4$; denote by w_2 and w_3 the points of intersection of the circle $|w| = |w_1| - 1/4$ with the rays.

In the 2nd case denote by D the domain bounded by the same rays and by the smaller arcs of the circles $|w| = |w_1|$ and $|w| = |w_1| + 1/4$, and by w_2, w_3 the points of intersection of the same rays with the circle $|w| = |w_1| + 1/4$.

In the 3rd case, when $w_1 = 0$, the domain D will be the sector formed by the arc of the circle $|w| = 1/4$ enclosed between two arbitrary rays issuing from the origin, between which the angle is sufficiently small, while w_2, w_3 are the points of intersection of the circle $|w| = 1/4$ with these rays.

Let $\mu(z)$ be a function realizing a conformal mapping of the disk K onto the domain $\overline{D}$ with the additional conditions: the points z_i go into the points w_i $(i = 1, 2)$. Such a function exists by the Riemann theorem, is analytic inside the disk K, and is continuous in the closed disk.

Construct the function

$$\nu(z) = \begin{cases} \mu(z), & z \in \overline{K}, \\ 0, & z \notin \overline{K}. \end{cases}$$

It is then proved that the function $g(z) = f_0(z) - \nu(z)$ will be the desired one. This assertion is proved.

Remark. Analogously to Theorem 1, using part a) of Lemma 2 and part b) for $Y \neq Z$, and also the separability of the space L_1/H_0^1 (L'/H_0'), we obtain Theorem 3.

Theorem 3. *The space L_1/H_0^1 (L'/H_0') is not isomorphic to any subspace of a separable Banach space X^* conjugate to a Banach space X, nor to any subspace of a Banach space with an unconditional basis.*

In conclusion the authors express their gratitude to A. Pelczynski for posing the problem.

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Note: Figure translations are in progress. See original paper for figures.