APPROXIMATION OF DYNAMIC CHARACTERISTICS IN THE CLASS OF GENERALIZED FUNCTIONS

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Abstract

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APPROXIMATION OF DYNAMIC CHARACTERISTICS IN THE CLASS OF GENERALIZED FUNCTIONS

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Let $y_o(t)$ and $y_{\mathrm{m}}(t)$ be the values of the outputs of the object and of the model, respectively. For real objects $x(t)$ is a finite function, i.e., one that vanishes outside a certain finite interval $[t_0, t_1]$. Then, as is known ([1]), the relation between the output and the input of the object has the form

$$y_o(t) = \begin{cases} 0, & t < t_0, \\ \int_{t_0}^{t} x(u) k_o(t-u)\, du, & t_0 \le t \le t_1, \\ \int_{t_0}^{t_1} x(u) k_o(t-u)\, du, & t > t_1. \end{cases} \tag{1}$$

An analogous relation (2) is valid for $y_{\mathrm{m}}(t)$. Applying the direct and inverse Fourier transforms to relations (1) and estimating $|y_o(t) - y_{\mathrm{m}}(t)|$, we obtain

$$\begin{aligned} |y_o(t) - y_{\mathrm{m}}(t)| &= \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} [\Phi_o(i\omega) - \Phi_{\mathrm{m}}(i\omega)] X(i\omega) e^{i\omega t}\, d\omega \right| \\ &\le \frac{\max\limits_{t_0 \le t \le t_1} |x(t)| (t_1 - t_0)}{2\pi} \int_{-L}^{L} |\Phi_o(i\omega) - \Phi_{\mathrm{m}}(i\omega)|\, d\omega \\ &+ \frac{1}{2\pi} \left| \int_{-\infty}^{\infty'} [\Phi_o(i\omega) - \Phi_{\mathrm{m}}(i\omega)] X(i\omega) e^{i\omega t}\, d\omega \right|, \end{aligned} \tag{2}$$

where $\Phi_o(i\omega)$ and $\Phi_{\mathrm{m}}(i\omega)$ are the transfer functions of the object and the model; $[-L, L]$ is a finite interval on the axis (its choice will be discussed below);

$$\int_{-\infty}^{\infty'}$$

is the integral over the remaining part of the axis.

Let us first estimate the integral over the finite part of the axis

$$I_1 = \int_{-L}^{L} |\Phi_o(i\omega) - \Phi_{\mathrm{m}}(i\omega)|\, d\omega. \tag{3}$$

The integrand is the error of approximation of the transfer function of the object on the interval $[-L, L]$ by means of the interpolation polynomial ([2])

$$u_n^+(\Phi,\omega) = \sum_{k=0}^{n} a_k \left(\frac{\omega + i\alpha}{\omega - i\alpha}\right)^k, \qquad a_k = \frac{1}{n+1}\sum_{j=0}^{n} \Phi(i\omega_j)\left(\frac{\omega_j + i\alpha}{\omega_j - i\alpha}\right)^{-k}, \tag{4}$$

$$\omega_j = \alpha \operatorname{ctg}\frac{\theta_j}{2}, \qquad \theta_j = \frac{\pi}{n+1}(2j+1), \qquad -\pi \le \theta_j \le \pi, \qquad \alpha > 0.$$

Make the change of variables

$$\omega = i\alpha\frac{\tau+1}{\tau-1} = \alpha \operatorname{ctg}\frac{\theta}{2}, \qquad \tau = \frac{\omega + i\alpha}{\omega - i\alpha} = e^{i\theta}, \qquad -\pi \leqslant \theta \leqslant \pi,$$

and obtain an estimate for the approximation of the function $\varphi(\tau) = \Phi\left(i\alpha\frac{\tau+1}{\tau-1}\right)$ on the part of the unit circle γ' that does not include a certain neighborhood of the singular point $\tau = 1$.

Lemma 1. The estimate

$$|u_n^+(\varphi,\tau)| \leqslant \begin{cases} \left[\dfrac{4}{\pi} + \dfrac{2}{\pi}\ln\left(\dfrac{2}{\pi}(n+1)\right)\right] \max\limits_{\tau\in\gamma'} |\varphi(\tau)| & \text{for even } n, \\ \left[\dfrac{2}{\pi} + 1 + \dfrac{1}{\pi}\ln\left(\dfrac{2}{\pi}(n+1)\right)\right] \max\limits_{\tau\in\gamma'} |\varphi(\tau)| & \text{for odd } n, \end{cases} \tag{5}$$

(6)

holds, where the estimate (5) cannot be improved in the sense of its attainability.

The proof of estimates (5) and (6) follows easily from relation (5.5) of work ([3]).

Consider an estimate of the accuracy of approximation of the function $\varphi(\tau)$, specified on γ', by means of partial sums in Faber polynomials. For this purpose we construct the function $\tau = \psi(w)$, which conformally maps the exterior of the unit disk onto the exterior of the arc γ' ([4]). It has the form

$$\tau = \psi(w) = w\,\frac{iw\cos\alpha/2 - 1}{w + i\cos\alpha/2}, \qquad \tau \in \gamma', \tag{7}$$

where α is the angle between the radius drawn to the endpoint of the arc γ' and the positive direction of the abscissa axis; $w = \Phi(\tau)$ is the function inverse to $\psi(w)$.

Let $\chi^+(w)$ be the function analytic inside the unit disk ($|w| \leqslant 1$), obtained from the function $\chi(w) = \varphi[\psi(w)]$ by the Sokhotskii formula ([5])

$$\chi^+(w) = \frac{1}{2}\chi(w) + \frac{1}{2\pi i}\int_\gamma \frac{\chi(u)}{u-w}\,du, \qquad P_n(\tau) = \sum_{k=0}^n a_k \Phi_k(\tau), \tag{8}$$

where $\Phi_k(\tau)$ are the Faber polynomials for the continuum γ', and the a_k are such that

$$\left|\chi^+(w) - \sum_{k=0}^n a_k w^k\right| = \inf_{a_k}\left|\chi^+(w) - \sum_{k=0}^n a_k w^k\right| = \rho_n[\chi^+(w)].$$

In the notation introduced above, the following theorem holds:

Theorem 1. The estimate

$$\max_{\tau\in\gamma'}\left|\varphi(\tau) - \sum_{k=0}^n a_k\Phi_k(\tau)\right| \leqslant 3\rho_n[\chi^+(w)] \tag{9}$$

is valid.

Proof. For the Faber polynomials the relation ([5])

$$\Phi_k(\tau) = \frac{w_1^k + w_2^k}{2} + \frac{1}{2\pi i}\int_{L_1}\frac{[\Phi(\zeta)]^k\,d\zeta}{\zeta-\tau}, \qquad \tau\in L_1, \quad \psi(w_1) = \psi(w_2) = \tau, \tag{10}$$

holds, where L_1 is a closed curve consisting of two banks of γ'. Further,

$$\varphi(\tau) - \sum_{k=0}^n a_k\Phi_k(\tau) = \frac{1}{2}\left[\chi(w_1) - \sum_{k=0}^n a_k w_1^k\right] + \frac{1}{2}\left[\chi(w_2) - \sum_{k=0}^n a_k w_2^k\right]$$

$$+\frac{1}{2\pi i}\int_{|w|=1}\frac{\chi(w) - \sum_{k=0}^n a_k w^k}{\psi(w)-\psi(\tau)}\,\psi'(w)\,dw.$$

Using the relation

$$\chi(w) = \chi^+(w) - \chi^-(w)$$

and taking into account that

$$\frac{1}{2\pi i}\int_{|w|=1}\frac{\chi^-(w)\psi'(w)}{\psi(w)-\psi(\tau)}\,dw=-\frac{1}{2}\left[\chi^-(w_1)-\chi^-(w_2)\right],$$

$$\frac{\psi'(w)}{\psi(w)-\psi(\tau)}=\frac{1}{w-\tau}-\frac{1}{w+i\cos\alpha/2}+\frac{1}{w-\tau^*},\qquad \tau^*=\frac{1-i\tau\cos\alpha/2}{\tau+i\cos\alpha/2},$$

we arrive at the proof of estimate (9).

The obtained estimate (9) for the accuracy of approximation of the function $\varphi(\tau)$, specified on γ', by means of partial sums of Faber polynomials is analogous to the results given in works ([6–9]). The method of proving estimate (9) differs from the methods used in works ([6–8]).

Theorem 2. The estimate

$$\max_{\tau\in\gamma'}|\varphi(\tau)-u_n^+(\varphi,\tau)|\le \begin{cases}3\left[1+\frac{4}{\pi}+\frac{2}{\pi}\ln\left(\frac{2}{\pi}(n+1)\right)\right]\rho_n[\chi^+(w)] & \text{for } n \text{ even},\\ 3\left[2+\frac{2}{\pi}+\frac{1}{\pi}\ln\left(\frac{2}{\pi}(n+1)\right)\right]\rho_n[\chi^+(w)] & \text{for } n \text{ odd}.\end{cases}\tag{11}$$

The proof of Theorem 2 follows from Theorem 1 and Lemma 1. As a result, we obtain an estimate for I_1 (for example, for n even)

$$I_1\le 6L\left[1+\frac{4}{\pi}+\frac{2}{\pi}\ln\left(\frac{2}{\pi}(n+1)\right)\right]\rho_n[\chi^+(w)].\tag{13}$$

Let us now estimate the integral over the remaining part of the axis

$$I_2=\left|\int_{-\infty}^{\infty\,\prime}[\Phi_0(i\omega)-\Phi_M(i\omega)]X(i\omega)e^{i\omega t}\,d\omega\right|.\tag{14}$$

We shall carry out the estimate under the following restrictions on $X(i\omega)$ and $\Phi_0(i\omega)$:

$$\begin{aligned}&1.\quad X(i\omega)=\frac{1}{\omega^{k+1}}\sum_{j=1}^{r}c_je^{i\omega\alpha_j}+X_1(i\omega),\qquad |X_1(i\omega)|\le\frac{c}{|\omega|^p},\\ &2.\quad \Phi_0(i\omega)=\omega^k\sum_{s=1}^{l}d_se^{i\omega\beta_s}+\Phi_1(i\omega),\qquad |\Phi_1(i\omega)|\le\bar{c}\,|\omega^{k-1}|,\end{aligned}\tag{15}$$

$$k \geq 0, \qquad p > k+1, \qquad c_j \geq 0, \qquad d_s \geq 0,$$

c_j, d_s are real, and c and $\bar{c}$ are absolute constants.

Further,

$$I_2 \leq \left| \int_{-\infty}^{\infty'} \Phi_0(i\omega) X(i\omega) e^{i\omega t}\, d\omega \right| + \left| \int_{-\infty}^{\infty'} \Phi_M(i\omega) X(i\omega) \cos\omega t\, d\omega \right| +$$

$$+ \left| \int_{-\infty}^{\infty'} \Phi_M(i\omega) X(i\omega) \sin\omega t\, d\omega \right|.$$

Lemma 2. The estimate

$$\left| \int_{L}^{\infty'} \Phi_M(i\omega) X(i\omega) \cos\omega t\, d\omega \right|$$
$$\leq \frac{4}{L^{k+1}} \sum_{j=1}^{r} c_j \left[\frac{2\pi |A_0|}{t+\alpha_j} + \frac{c(n)n}{k+1} \right] + \frac{2c}{L^{p-1}} \left[\frac{|A_0|}{n-1} + \frac{2c(n)n}{(p+1)L^2} \right], \tag{16}$$

where A_0 is the first coefficient of the expansion of $\Phi_M(i\omega)$ into partial fractions.

$$\Phi_M(i\omega) = \sum_{k=0}^{n} \alpha_k \left(\frac{i\omega - \alpha}{i\omega + \alpha} \right)^k = A_0 + \frac{A_1}{i\omega + \alpha} + \cdots + \frac{A_n}{(i\omega+\alpha)^n}, \tag{17}$$

$$c(n) = \left[\frac{4}{\pi} + \frac{2}{\pi} \ln \left(\frac{2}{\pi}(n+1) \right) \right] \max_j |\varphi(\tau_j)|.$$

Proof. The estimate

$$\left| \int_{L}^{\infty} \frac{\cos\omega t}{\omega^m}\, d\omega \right| \leq \frac{2\pi}{L^m t}, \quad m \geq 1. \tag{18}$$

is valid.

From relation (17) we have

$$\Phi_M - A_0 = \frac{A_1}{i\omega+\alpha} + \frac{A_2}{(i\omega+\alpha)^2} + \cdots + \frac{A_n}{(i\omega+\alpha)^n} = u_n^+(\varphi, \tau) - u_n^+(\varphi, 1).$$

Further, using Bernstein' s inequality, we obtain

$$|u_n^+(\varphi,\tau) - u_n^+(\varphi,1)| \le \max_{\tau} |[u_n^+(\varphi,\tau)]'|\,|\tau - 1| \le c(n)n \cdot 2/|\omega - i|,$$

$$|\Phi_M(i\omega) - A_0| \le |A_0| + 2c(n)n/\sqrt{1+\omega^2}. \tag{19}$$

Using relations (18) and (19), we arrive at the proof of relation (16).

Theorem 3. If relation (15) is valid, then

$$\begin{aligned} I_2 \le \frac{2}{L} &\left(4\pi \sum_{j=1}^{r} \sum_{s=1}^{l} \frac{c_j d_s}{t + \alpha_j + \beta_s} + \bar{c} \sum_{j=1}^{r} c_j\right) \\ &+ \frac{2c}{L^{p-k-1}} \left(\frac{\sum_{s=1}^{l} d_s}{k - p + 1} + \frac{\bar{c}}{(k-p)L}\right) \\ &+ \frac{4}{L^{k+1}} \sum_{j=1}^{r} c_j \left[\frac{2\pi |A_0|}{t + \alpha_j} + \frac{c(n)n}{k+1}\right] \\ &+ \frac{2c}{L^{p-1}} \left[\frac{|A_0|}{p-1} + \frac{2nc(n)}{(p+1)L^2}\right]. \end{aligned} \tag{20}$$

The proof follows from relations (16)–(19).

As a result, under assumption (15), taking into account relations (2), (13), and (20), we obtain the following theorem.

Theorem 4. The estimate

$$\begin{aligned} |y_0(t) - y_M(t)| \le{}& 6L[1 + c(n)]\rho_n[\chi^+(w)] \\ &+ \frac{2}{L} \left(4\pi \sum_{j=1}^{r} \sum_{s=1}^{l} \frac{c_j d_s}{t + \alpha_j + \beta_s} + \bar{c} \sum_{j=1}^{r} c_j\right) \\ &+ \frac{2c}{L^{p-k-1}} \left(\frac{\sum_{s=1}^{l} d_s}{k - p + 1} + \frac{\bar{c}}{(k-p)L}\right) \\ &+ \frac{4}{L^{k+1}} \sum_{j=1}^{r} c_j \left[\frac{2\pi |A_0|}{t + \alpha_j} + \frac{c(n)n}{k+1}\right] \\ &+ \frac{2c}{L^{p-1}} \left[\frac{|A_0|}{p-1} + \frac{2c(n)n}{(p+1)L^2}\right]. \end{aligned} \tag{21}$$

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REFERENCES

1. V. V. Solodovnikov, A. S. Uskov, *Statistical Analysis of Control Objects*, Moscow, 1960.

2. V. K. Zadiraka, *Transactions of the Seminar on Algorithmization of Production Processes*, vol. 8, Kiev, 1965, p. 4.

3. V. V. Ivanov, V. K. Zadiraka, *Interdepartmental Scientific Collection. Computational Mathematics*, vol. 2, Kiev, 1966, p. 3.

4. M. A. Lavrent' ev, B. V. Shabat, *Methods of the Theory of Functions of a Complex Variable*, "Nauka," Moscow, 1965, pp. 147–148.

5. A. I. Markushevich, *Theory of Analytic Functions*, Moscow, 1950, pp. 418–423.

6. S. N. Mergelyan, *UMN*, 7, 2 (48), § 1, 1952.

7. S. Ya. Al' per, V. V. Ivanov, *DAN*, 40, No. 3, 325 (1953).

8. B. A. Vostretsova, A. V. Ignat' eva, *Scientific Notes of the Moscow Regional Pedagogical Institute, Proceedings of the Department of Mathematics*, 7, issue 4 (1957).

9. N. K. Bari, *Scientific Notes of Moscow University*, issue 181, Mathematics, 8 (1956).

Note: Figure translations are in progress. See original paper for figures.