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Abstract

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MATHEMATICS

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ON CERTAIN CLASSES OF MAPPINGS IN THE PLANE

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In the present note classes of mappings are introduced that are generalizations of the class \widetilde{BL}_k , to the consideration of which the monographs of G. D. Suvorov ⁽¹⁾ and, in a somewhat less general situation, J. Lehto and K. I. Virtanen ⁽²⁾ are devoted. Further, following the method developed by G. D. Suvorov for studying mappings of the class \widetilde{BL}_k , two-sided estimates are established for the distortion of relative distance in closed domains, which makes it possible to transfer the greater part of the results from ⁽¹⁾ and partially from ⁽²⁾ to mappings of the classes considered. Some results are also given that are new even for mappings of the class \widetilde{BL}_k .

1. Let $w = f(z) = u(x, y) + iv(x, y)$ be a continuous mapping of a domain A of the plane $z = x + iy$, with the spherical metric (r) introduced in it, onto a domain B of the plane $w = u + iv$ with metric (R). We shall say that $w = f(z)$ belongs to the class \widetilde{BL}_k^Φ in the domain A if $f(z)$ has in A generalized partial derivatives u_x, u_y, v_x, v_y in the sense of S. L. Sobolev, for which

$$I_\Phi(f) = \int_A \Phi(\|f'(z)\|) dA \leq K < \infty, \quad (1)$$

where $\Phi(t)$ is a nonnegative, convex downward ($\Phi'' > 0$) function, defined for all $t \in [0, \infty)$ and such that

$$\int_0^\infty \frac{\Phi(\tau)}{\tau^3} d\tau = \infty, \quad (2)$$

and by $\|f'(z)\|$ is denoted the expression

$$(u_x^2 + u_y^2 + v_x^2 + v_y^2)^{1/2} / (1 + |f(z)|^2 / R^2).$$

For $\Phi(\tau) = \tau^2$ this class of mappings coincides with the class \widetilde{BL}_k considered in ⁽¹⁾. At the same time, mappings of the classes \widetilde{BL}_k^Φ encompass a significantly broader part of mappings, as may be judged, for example, from Remarks 1 and 2 to Theorem 1. It is not difficult to see that there exist such Φ_1 and Φ_2 , satisfying condition (2), for which the inclusions

$$\widetilde{BL}_k^{\Phi_1} \supset \widetilde{BL}_k \supset \widetilde{BL}_k^{\Phi_2}$$

hold.

For simplicity of exposition we shall restrict ourselves here to the case when

$$\widetilde{BL}_k^\Phi \supset BL_k.$$

2. Let z_1, z_2 be arbitrary points of a simply connected domain $A \subset E^2$ containing the origin. Following G. D. Suvorov ⁽¹⁾, p. 50, we define the relative distance between the points z_1, z_2 in the domain A by

$$\rho_A(z_1, z_2; r) = \min[\rho_1(z_1, z_2), \rho_2(z_1, z_2)], \quad (3)$$

where $\rho_1(z_1, z_2)$ is the greatest lower bound of the diameters, taken in the spherical metric (r), of curves contained in A and joining the points z_1, z_2 , and $\rho_2(z_1, z_2)$ is the greatest lower bound of the spherical diameters of curves separating points z_1, z_2 from the origin. Complete the domain A with respect to the relative distance. The “boundary” elements adjoined in this process are identified with the prime ends of K. Carathéodory. The resulting space will be denoted by \widetilde{A} .

The following theorem establishes the order of equicontinuity and equiopenness (see, for example, ⁽¹⁾, p. 94) of topological mappings of the class \widetilde{BL}_k^Φ in closed domains.

Theorem 1. Let $w = f(z)$ be a homeomorphic mapping of class \widetilde{BL}_k^Φ of a domain $A \in E^2$ onto a domain $B \in E^2$, normalized by the conditions $f(0) = 0$, $f(z_0) = w_0$.

Then $f(z)$ extends to a continuous mapping of \widetilde{A} onto \widetilde{B} , and for any two points $z', z'' \in \widetilde{A}$ such that $\rho_{\widetilde{A}}(z', z''; r) \leq \varepsilon_1$, the inequality

$$\rho_{\widetilde{B}}[f(z'), f(z''); R] \leq \varphi_\Phi[\rho_{\widetilde{A}}(z', z''; r)] \quad (4)$$

holds, where

$$\varphi_\Phi(t) = \left[C(k, \Phi) \int_1^{1/t} \frac{\Phi(\tau)}{\tau^3} d\tau \right]^{-1/2}. \quad (5)$$

If, moreover, the inverse mapping $z = f^{-1}(w)$ belongs to the class \widetilde{BL}_k^Φ in the domain B , then the lower estimate

$$\varphi_\Phi^{-1}[\rho_{\widetilde{A}}(z', z''; r)] \leq \rho_{\widetilde{B}}[f(z'), f(z''); R], \quad (6)$$

also holds, valid for all $z', z'' \in \widetilde{A}$ for which $\rho_{\widetilde{A}}(z', z''; r) \leq \varepsilon_2$. The constants $\varepsilon_1, \varepsilon_2$ depend on $\Phi, \rho_{\widetilde{A}}(0, \partial A; r), \rho_{\widetilde{B}}(0, \partial B; R)$, and on the normalization conditions.

Remark 1. Estimates (4), (6) are sharp in the sense of order.

Remark 2. Since the rate of divergence of the integral (2) can, by a suitable choice of $\Phi(\tau)$, be made arbitrarily small, it is evident that the classes \widetilde{BL}_k^Φ may have arbitrarily low order of equicontinuity. This circumstance indicates the considerable breadth of the classes \widetilde{BL}_k^Φ in comparison with the class BL_k .

The existence of estimates (4), (6) makes it possible to carry over a large part of the results from ⁽¹⁾ to mappings of the classes \widetilde{BL}_k^Φ . As an example, one may mention the theorem on boundary correspondence by prime ends which follows directly from Theorem 1. The two-sided equicontinuous estimates of the distortion of spherical (Euclidean) distance inside domains, which follow easily from (4), (6), give an analogue of Carathéodory's theorem on convergence to the kernel of a sequence of domains (see ⁽¹⁾). The two-sided equicontinuous estimates of the distortion of relative distance in closed domains make it possible to extend the theory of prime ends of a sequence of domains converging to a nondegenerate kernel ⁽¹⁾ to the case of mappings $f, f^{-1} \in \widetilde{BL}_k^\Phi$, with all the consequences flowing from this; among them we mention here only the theorem on the stability of conformal mappings in closed domains ⁽³⁾. In addition, on the basis of inequalities (4), (6) one can obtain various estimates of the distortion of boundary arcs and of the areas of boundary rings (see ⁽⁴⁾).

3. The following assertions are new also for mappings of the class \widetilde{BL}_k .

Theorem 2. Let A be a domain in E^2 whose boundary contains a one-point component $z_0 \in \partial A$. Let $f(z)$ be an open mapping of class \widetilde{BL}_k^Φ of the domain A into E^2 .

Then $f(z)$ is continuous in the spherical metric at the point $z_0 \in \partial A$.

This theorem indicates the fact that open mappings of the class \widetilde{BL}_k^Φ may have isolated singularities only of the pole type. As a consequence of it we obtain an analogue of Liouville's theorem, well known in the theory of analytic functions.

Theorem 3. There is no open mapping of the class \widetilde{BL}_k^Φ of the entire plane E^2 onto a proper part of it.

The following theorem gives a simple criterion for normality, in the sense of P. Montel ⁽⁵⁾, of a family of mappings of the class \widetilde{BL}_k^Φ .

Theorem 4. A family of F -open mappings $f(z)$ of the class \widetilde{BL}_k^Φ in a domain $A \subset E^2$, not assuming in this domain one finite value $w_0 \in E^2$, is normal.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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