HARNACK' S INEQUALITY FOR ELLIPTIC EQUATIONS OF THE SECOND ORDER OF CORDES TYPE

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Abstract

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HARNACK' S INEQUALITY FOR ELLIPTIC EQUATIONS OF THE SECOND ORDER OF CORDES TYPE

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In this note we shall consider the linear elliptic equation of the second order

$$Lu \equiv \sum_{i,k=1}^{n} a_{ik}(x)\frac{\partial^2 u}{\partial x_i \partial x_k} = 0. \tag{1}$$

We shall call such an equation an equation of **Cordes type** in some domain D, if there exists a positive constant $s < n$ such that

$$L(1/|x-x^0|^s) \geq 0, \tag{2}$$

$x \in D$, x^0 being any fixed point of R_n.

Let, for equation (1), the inequality

$$\sum_{i,k=1}^{n} a_{ik}\xi_i\xi_k \geq \alpha|\xi|^2 \tag{3}$$

be satisfied.

It is easy to verify that if $\sup \sum_{i=1}^{n} a_{ii}/\alpha < n+2$, the equation is of Cordes type.

We shall assume that equation (1) is of Cordes type. Nothing more will be assumed about its coefficients, including measurability.

By a solution of the equation we shall mean a classical solution—a continuous function having the derivatives entering into the equation and reducing it to an identity.

For an equation of Cordes type, Harnack' s inequality is proved. We shall denote by Q_R the ball of radius R with center at the origin of coordinates.

Lemma 1. Let a domain D be situated in the ball Q_4, have boundary points on the surface S_4 of the ball Q_4, and intersect the ball Q_1. Denote by Γ that part of the boundary of D which lies strictly inside Q_4. Let H be the intersection of the complement of the domain D with the ball Q_1. Let an equation (1) of Cordes type be defined in D, and let $u(x)$ be its solution, positive in D, continuous in $\overline{D}$, and vanishing on Γ. Then

$$\max_{\overline{D}} u \geq (1 + \eta \text{ mes } H) \max_{\overline{D} \cap Q_1} u,$$

where $\eta > 0$ is a constant depending on the constant s in inequality (2) and on the dimension n of the space.

Proof. Put

$$\int_{Q_1} \frac{dx}{|x|^s} = K. \tag{4}$$

Then for any $y \in R_n$

$$U(y) = \frac{1}{K} \int_H \frac{dx}{|x-y|^s} \leqslant \frac{1}{K} \int_{Q_1} \frac{dx}{|x-y|^s} \leqslant \frac{1}{K} \int_{Q_1} \frac{dx}{|x|^s} = 1. \tag{5}$$

Construct the auxiliary function

$$V(x) = m \left(1 - U(x) + \frac{\text{mes } H}{K \cdot 3^s}\right), \qquad \text{where } m = \max_{\overline{D}} u.$$

We have $LV \geqslant 0$ outside H. Further, on the boundary of the domain D the function V is not less than u. Indeed, the boundary of D consists of Γ (where $u = 0$, while V is positive) and of points belonging to S_4. But $u|_{S_4} \leqslant m$, and

$$U\big|_{S_4} = \frac{1}{K} \int_H \frac{dy}{|x-y|^s}\bigg|_{S_4} \leqslant \frac{\text{mes } H}{K \cdot 3^s},$$

and therefore $V|_{S_4} \geqslant m$. Hence, by the maximum principle, $V \geqslant u$ in D, and

$$\max_{\overline{D} \cap Q_1} u \leqslant \max_{\overline{D} \cap Q_1} V \leqslant m \left(1 + \frac{\text{mes } H}{K \cdot 3^s} - \min_{Q_1} U\right) \leqslant m \left[1 + \frac{1}{K} \left(\frac{1}{3^s} - \frac{1}{2^s}\right) \text{mes } H\right],$$

which proves the lemma, since from (4) it follows that K depends on s and n.

With the aid of Lemma 1 we prove

Lemma 2. Let $R > 0$ be arbitrary. Let D be a domain lying inside Q_{2R}, intersecting Q_R, and having boundary points on the boundary of the ball Q_{2R}. Denote by Γ that part of the boundary of the domain D which is situated strictly inside Q_{2R}. Let $u(x)$ be a solution of equation (1) of Cordes type, positive in D, continuous in $\overline{D}$, and vanishing on Γ.

There exists a constant $\delta > 0$, depending on s and n, such that from the inequality $\operatorname{mes} D / \operatorname{mes} Q_R < \delta$ it follows that

$$\max_{\overline{D}} u \geqslant 2^{n+1} \max_{\overline{D} \cap Q_R} u.$$

Theorem 1 (Harnack inequality). Let equation (1) of Cordes type be defined in the ball Q_{16}. Let $u(x)$ be a solution of this equation, positive in the ball Q_{16}. Then

$$\max_{|x| \leqslant 1} u(x) / \min_{|x| \leqslant 1} u(x) < C,$$

where C is a constant depending on s and n.

We give here the main points of the proof. Put $\max_{|x| \leqslant 1} u(x) = m$ and $v(x) = \frac{2}{m} u(x)$. We have to prove that there exists a constant $\gamma > 0$, depending on s and n, such that

$$\min_{|x| \leqslant 1} v(x) > \gamma. \tag{6}$$

Denote by D_1 the set of points $x \in Q_4$ where $v(x) > 1$. We consider separately two cases: 1) $\operatorname{mes} D_1 > \delta \omega_n / 4^n$, where δ is the constant of Lemma 2 and ω_n is the volume of the unit n-dimensional sphere, and 2) $\operatorname{mes} D_1 \leqslant \delta \omega_n / 4^n$.

In the first case the desired inequality (6) is easily obtained by applying Lemma 1 to the function $w(x) = 1 - v(x/4)$. We shall consider the second case.

Denote by S_{r_1, r_2} the spherical layer $r_1 \leqslant |x| \leqslant r_2$. Let ρ_1 be the largest of the numbers ρ, $0 < \rho < 1$, such that

$$\operatorname{mes}(D_1 \cap S_{1,\, 1+\rho}) \geqslant \delta \omega_n (\rho/4)^n. \tag{7}$$

Since for ρ sufficiently close to zero this inequality is satisfied, while for $\rho = 1$ the reverse inequality holds, such a ρ_1 exists. Put $G_1 = D_1 \cap S_{1,\, 1+\rho_1}$. Then $\operatorname{mes} G_1 = \delta \omega_n (\rho/4)^n$.

Consider the function $w_1(x) = v(x) - 1$. It is positive on G_1, nonpositive on $S_{1,\, 1+\rho_1} \setminus G_1$, and the intersection of G_1 with the sphere $|x| = 1 + \rho_1/2$ contains a point x^1 where $w_1(x^1) > 1$. Denote by Q^1 the ball of radius $\rho_1/2$ with center at the point x^1, and by $Q^1_{1/2}$ the concentric ball of half the radius. Denote by D^1 that component of the intersection $G_1 \cap Q^1$ which contains the point x^1. Applying Lemma 2 to Q^1, $Q^1_{1/2}$, and D^1, and then the maximum principle, we find

$$\max_{|x|=1+\rho_1} v(x) \geq \max_{\overline{Q}^1} v > \max_{\overline{Q}^1} w \geq 2 \cdot 2^n.$$

Now denote by D_2 the set of those $x \in Q_4$ where $v > 2^n$. Let ρ_2 be the largest of the numbers ρ, $0 < \rho < 1$, such that

$$\operatorname{mes}(D_2 \cap S_{1+\rho_1,\,1+\rho_1+\rho}) \geq \delta\omega_n(\rho/4)^n.$$

Put $G_2 = D_2 \cap S_{1+\rho_1,\,1+\rho_1+\rho_2}$, $w_2(x) = v(x) - 2^n$. Then $\operatorname{mes} G_2 = \delta\omega_n(\rho/4)^n$. We have $w_2(x) > 0$ on G_2, $w_2(x) \leq 0$ on $S_{1+\rho_1,\,1+\rho_1+\rho_2}$. At the intersection of G_2 with the sphere $|x| = 1+\rho_1+\rho_2$ there is a point x^2 where $w_2(x^2) > 2^n$. Denoting by Q^2 and $Q^2_{1/2}$ the balls with center at x^2 of radii $\rho_2/2$ and $\rho_2/4$, respectively, and by D^2 that component of the intersection $G_2 \cap Q^2$ which contains x^2, and applying Lemma 2 and the maximum principle, we find

$$\max_{|x|=1+\rho_1+\rho_2} v(x) \geq \max_{\overline{Q}^2} v > \max_{\overline{Q}^2} w_2 \geq 2 \cdot 2^{2n}.$$

We shall continue this process until, for the first time, it becomes true that $\rho_1 + \rho_2 + \dots + \rho_k \geq 1$. This must necessarily occur at a finite step, since otherwise the function $v(x)$ would grow to infinity.

Thus, suppose k steps have been carried out, and $\rho_1 + \dots + \rho_k \geq 1$. For each i, $i = 1, \dots, k$, there is a set G_i such that

$$\operatorname{mes} G_i = \delta\omega_n(\rho_i/4)^n, \qquad v(x) > 2^{n(i-1)} \quad \text{on } G_i.$$

Among the numbers $\rho_1, \dots, \rho_k$ there is one ρ_{i_0} such that $\rho_{i_0} > 1/2^{i_0}$. Therefore, in the spherical layer $Q_3 \setminus Q_1$ there is a set G_{i_0} such that

$$\operatorname{mes} G_{i_0} > \frac{\delta\omega_n}{4^n}\frac{1}{2^{ni_0}}, \qquad v > 2^{n(i-1)} \quad \text{on } G_{i_0}.$$

Put

$$W(x) = \frac{2^{ni_0}}{2^{nK}} \int_{G_{i_0}} \frac{dy}{|x-y|^s} - \frac{2^{ni_0}}{K \cdot 12^s} \operatorname{mes} G_{i_0}, \qquad K = \int_{Q_3} \frac{dx}{|x|^s}.$$

Consider on the set $\Omega = Q_{16} \setminus G_{i_0}$ the functions $W(x)$ and $v(x)$. The boundary Ω consists of the surface S_{16} of the ball Q_{16} and of the boundary Γ_{i_0} of the set G_{i_0}. We have

$$v(x)\big|_{\Gamma_{i_0}} \geq 2^{n(i-1)}; \qquad W(x)\big|_{\Gamma_{i_0}} \leq 2^{n(i_0-1)};$$

$$v(x)\big|_{S_{16}} \geq 0; \qquad W(x)\big|_{S_{16}} < \frac{2^{ni_0}}{2^{nK}} \left(\max_{\substack{|x|=16 \\ y\in Q_3}} \frac{1}{|x-y|^s} - \frac{1}{12^s} \right) \operatorname{mes} G_{i_0} < 0.$$

Consequently, by the maximum principle, $v \geq W$ in Ω. Therefore

$$v(x)\big|_{Q_1} \geq W(x)\big|_{Q_1} > \frac{2^{ni_0}}{2^{nK}} \left(\min_{\substack{x\in Q_1 \\ y\in Q_3}} \frac{1}{|x-y|^s} - \frac{1}{12^s} \right) \operatorname{mes} G_{i_0} > \frac{\delta\omega_n}{8^{nK}} \left(\frac{1}{4^s} - \frac{1}{12^s} \right) = \gamma,$$

where γ, thus, depends on s and n.

Remark. Since equation (1) is unchanged under a similarity transformation, in the statement of the theorem the absolute values of the radii of the outer and inner balls (16 and 1) are not essential; what matters only is

to their ratio; of course, this ratio can be made arbitrary by changing the constant C in the corresponding way. The same remark also applies to Theorem 2, proved below.

For the validity of Harnack' s inequality it is not necessary that equation (1) be of Cordes type in the whole ball Q_{16}. It is sufficient that it be of Cordes type in a layer adjacent to its boundary.

Theorem 2 (Harnack' s inequality). *Let an elliptic equation (1) be defined in the ball Q_{16}. Suppose that outside the ball Q_1 it is of Cordes type. Let $u(x)$ be a positive solution of this equation in Q_{16}. Then*

$$\max_{\overline{Q}1} u(x) / \min_{\overline{Q}1} u(x) < C,$$

where the constant C depends on s and n.

Proof. Denote by S_8 the sphere $|x| = 8$. Denote by k the least natural number such that for any two points a and b lying on S_8, one can find a chain of k points $x^1, \dots, x^k$, also lying on S_8, such that $x^1 = a$, $x^k = b$ and $|x^{i+1} - x^i| < 7/16$, $i = 1, \dots, k-1$.

The number k, obviously, depends on the dimension n of the space. We then have

$$\max_{x \in S_8} u(x) / \min_{x \in S_8} u(x) < C^{k-1},$$

where C is the constant in the Harnack inequality proved in the preceding theorem. Indeed, let the minimum on the sphere S_8 be attained at the point a, and the maximum at the point b. Construct the chain $a = x^1, x^2, \ldots, x^k = b$ mentioned above. Applying the proved inequality successively to balls of radius 7 with centers at the points x^i, $i = 1, \ldots, k-1$, we obtain $u(b) < C^{k-1} u(a)$.

It remains now for us to note that, by the maximum principle,

$$\max_{\overline{Q}1} u / \min_{\overline{Q}1} u \leq \max_{S_8} u / \min_{S_8} u.$$

Theorem 2 gives us the possibility of obtaining Liouville' s theorem for elliptic equation (1) under the assumption that it is of Cordes type in a neighborhood of infinity.

Theorem 3 (Liouville). *Let an elliptic equation (1) be defined in the entire space R_n and, outside a ball of some radius R, be of Cordes type. Let $u(x)$ be a solution of this equation, defined in the entire space and nonnegative. Then* $u(x) \equiv \text{const}$.

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Note: Figure translations are in progress. See original paper for figures.