SOLUTION OF NONLINEAR INTEGRAL EQUATIONS BY INTRODUCING A CONTINUOUS PARAMETER

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Abstract

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SOLUTION OF NONLINEAR INTEGRAL EQUATIONS BY INTRODUCING A CONTINUOUS PARAMETER

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Consider the nonlinear integral equation

$$u(x) = \int_a^b f(x, \xi, u(\xi))\, d\xi \tag{1}$$

in the Lipschitz space $C^{(L)}_{(a,b)}$, i.e., in the subspace of the space $C_{(a,b)}$ of functions $u(x)$ continuous on the interval (a, b), additionally satisfying the Lipschitz condition

$$|u(x_1) - u(x_2)| \leq L|x_1 - x_2|, \quad x_1, x_2 \in (a, b). \tag{2}$$

Denoting by L_u the greatest lower bound of all possible Lipschitz constants in condition (2), we introduce a norm in the space $C^{(L)}_{(a,b)}$ as follows:

$$\|u\| = \max_{a \leq x \leq b} |u(x)| + L_u. \tag{3}$$

The solution of nonlinear integral equations of this type is usually carried out by the method of successive approximations ([1]), which imposes very stringent requirements on the kernel of the equation.

Recently developed methods for solving nonlinear integral equations by averaging functional corrections ([2]) are sufficiently effective, but they also require restrictions on the Lipschitz constant and on the integral properties of the function $f(x, \xi, u)$ in (1).

In the present paper a method is proposed for solving equations of the form (1) by introducing a continuous parameter, close in idea to the work ([3]). The main content of the paper is the following theorem.

Theorem 1. Let there exist, inside a closed domain $D \in C^{(L)}_{(a,b)}$, a unique solution $u^*(x)$ of equation (1). In the case of nonuniqueness of the solution, it is assumed that D lies in the localization region of the solution, i.e., in the region

$$z(x) \le u(x) \le Z(x), \quad z, Z \in C^{(L)}_{(a,b)},$$

within which the solution is unique.

The function $f(x, \xi, u)$ in (1) is assumed to be continuous jointly in the variables and, in addition, to satisfy the following conditions in the domain D: a) twice continuously differentiable with respect to u; b) the Lipschitz condition with respect to x both for $f(x, \xi, u)$ and for $\partial f(x, \xi, u)/\partial u$.

Let, for any $u \in D$, the linear equation

$$u(x) - \int_a^b f'_u(x, \xi, \bar{u}(\xi))u(\xi)\, d\xi = 0 \tag{4}$$

have only the trivial solution.

Then there exists an $\varepsilon > 0$ such that, for any function $u_0(x)$ satisfying the condition

$$\|u_0(x) - u^*(x)\| < \varepsilon, \tag{5}$$

system of equations

$$v(x,t) - \int_a^b f'_u(x, \xi, u(\xi,t))v(\xi,t)\, d\xi + \int_a^b f(x, \xi, u(\xi,t))\, d\xi = u(x,t), \tag{6}$$

$$\partial u(\xi,t)/\partial t = v(\xi,t)$$

with the initial condition $u(x, 0) = u_0(x)$ has a unique solution in the domain D_1 and

$$\lim_{t \to +\infty} u(x,t) = u^*(x), \tag{7}$$

where $u^*(x)$ is a solution of equation (1), and D_1 is the direct product of the half-axis $0 \le t < +\infty$ and the domain D.

Convergence in relation (7) is understood in the sense of the metric of $C^{(L)}_{(a,b)}$.

For the proof of Theorem 1 we shall have to use some notions pertaining to Banach spaces.

It is not difficult to verify that the space $C^{(L)}_{(a,b)}$ with norm (3) is complete, linear, and normed, i.e., a Banach space; moreover, the integral operator

$$Au(x) \equiv u(x) - \int_a^b f(x, \xi, u(\xi))\, d\xi \tag{8}$$

will define a mapping of the space $C^{(L)}_{(a,b)}$ into itself.

The problem of solving equation (1) may now be regarded as the problem of solving the operator equation

$$Au = 0. \tag{9}$$

The assumptions of Theorem 1 allow us to use the idea of the work [4] on reducing the problem posed to the study, in a Banach space, of questions of existence and behavior as $t \to \infty$ of the solution of the differential equation

$$du/dt = -[A']^{-1} Au, \qquad u(0) = u_0, \tag{10}$$

which is a continuous analogue of Newton' s method.

We formulate the general theorem for Banach spaces from [4] which we intend to use.

Theorem 2. Let equation (9) have a unique solution u^* in some open domain D of an arbitrary Banach space U. Suppose that in D there exist $A'u$ and $A''u$ –the first and second Fréchet derivatives of the operator A–and that $A''u$ is bounded in a neighborhood of each point $u \in D$.

Suppose, further, that there exists an inverse operator $[A']^{-1}$, for which in D the inequality

$$\|[A']^{-1} v\| \leq B\|v\| \tag{11}$$

holds. Then in the domain D there exists a sphere S, $\|u - u^*\| < \varepsilon$, such that for any $u_0 \in S$ equation (10) has a solution on the interval $0 \leq t < +\infty$ and

$$\lim_{t\to\infty} u(t) = u^*.$$

For the proof of Theorem 1 we shall also need the following

Lemma. Let all the conditions of Theorem 1 be satisfied. Denote by $A'_{\bar{u}}$ the Fréchet derivative of the operator (8) with respect to the function $\bar{u}$ from the closed sphere $\|u - u^*\| \leq K$, belonging to the domain D.

Then, for all $\|v\| \leq 1$, the solutions of the equation

$$A'_{\bar{u}} u = v \tag{12}$$

are uniformly bounded in D, i.e.,

$$\|u\| < M. \tag{13}$$

The proof of this lemma can be obtained without difficulty by contradiction using Theorem 3 on p. 157 of [5], and we do not give it here.

Proof of Theorem 1. Assumptions a) and b) of Theorem 1 ensure the existence in the domain D of the first and second Fréchet derivatives of the operator (8). Condition (4) ensures the existence of the single inverse operator $[A'_u]^{-1}$. The lemma just proved means that inequality (11) holds. Thus all the hypotheses of Theorem 2 are satisfied, according to which there exists a solution of equation (10) that converges to the solution of the original equation (9). For operators of the form (8), equation (10) is written, as is easy to verify, in the form of system (6). Theorem 1 is proved.

For solving system (6) one may propose the following numerical scheme. Divide the domain $D_1(T) : a \le x \le b,\ 0 \le t \le T$ into n parts by segments parallel to the x-axis through points with coordinates $0, \tau_1, \tau_2, \dots, \tau_n = T$. Having specified an initial approximation $u(x,0) = u_0(x)$, satisfying condition (5) of Theorem 1, we can determine $v(x,0)$ from the linear integral equation—the first equation of system (5)

$$v(x,0) = \int_a^b f'_u(x,\xi,u_0(\xi))v_0(\xi,0)\,d\xi + \int_a^b f(x,\xi,u_0(\xi))\,d\xi - u_0(\xi).$$

After this, the difference analogue of the second equation of system (6) will allow us to determine the function $u(x,\tau_1)$ on the second layer for $t = \tau_1$

$$u(x,\tau_1) = u_0(x) + \tau_2 v(x,0).$$

In general, if the function $u(x,t)$ is known on the layer $t = \tau_k$, then to determine $v(x,\tau_k)$ one may solve, by any known method, the linear integral equation

$$v(x,\tau_k) = \int_a^b f'_u(x,\xi,u(\xi,\tau_k))v(\xi,\tau_k)\,d\xi + \int_a^b f(x,\xi,u(\xi,\tau_k))\,d\xi - u(x,\tau_k). \tag{14}$$

(This can be done, for example, by replacing the integral in (14) by an integral sum and passing to a linear algebraic system.) After this, the function $u(x,\tau_{k+1})$ is determined on the next layer $t = \tau_{k+1}$

$$u(x, \tau_{k+1}) = u(x, \tau_k) + \tau_{k+1} v(x, \tau_k). \tag{15}$$

In the numerical scheme described, it is not difficult to see a realization of Euler' s polygonal-line method for the operator equation (9). The convergence of this method as

$$\Delta\tau = \max_{1 \le k \le n} |\tau_k - \tau_{k-1}| \to 0$$

in a Banach space is known (see, for example, [4]). Choosing T sufficiently large, in accordance with Theorem 1 we should obtain stabilization of the functions $u(x, t)$ to the solution of the original equation (1).

As an example, the nonlinear integral equation taken from [2] was solved:

$$y(x) = x - 3 \int_0^1 \xi y^2(\xi)\, d\xi. \tag{16}$$

The equation has two solutions

$$y(x) = x - 1 \pm 1/\sqrt{2}.$$

The ordinary iteration method here does not converge to any definite limit.

In solving by the described method the system (14)–(15), constructed for equation (16), convergence to the exact solution of equation (16) was obtained rather quickly (in 3–5 steps in t), although as the initial

Fig. 1. 1—$y_0(x) = 1.1x + 0.2\sin 4\pi x$; 2—$y(x, 1)$; 3—$y(x, 2)$; 4— exact solution $y = x - 1 + 1/\sqrt{2}$; step in t, $\Delta t = 1.0$

Fig. 2. 1—$y_0(x) = 1.1x + 0.2\sin 4\pi x$; 2—$y(x, \frac{1}{2})$; 3—$y(x, 1)$; 4—$y(x, 1\frac{1}{2})$; 5—$y(x, 2\frac{1}{2})$; 6—$y = x - 1 + 1/\sqrt{2}$. Step in t, $\Delta t = 0.5$

approximation there was taken the nonlinear function $y_0(x) = a \sin 4\pi x + bx$. The results of the computations are shown in Figs. 1 and 2.

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Note: Figure translations are in progress. See original paper for figures.