ON SOME SYSTEMS OF SYMMETRIC FUNCTIONS

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 517.514+517.562.+517.564

MATHEMATICS

Corresponding Member of the Academy of Sciences of the USSR L. A. LYUSTERNIK

ON SOME SYSTEMS OF SYMMETRIC FUNCTIONS

We retain the notation of (1): x_i are Cartesian coordinates (c.c.) in E_n; e_j, $j = 0, 1, \dots, n$, are unit vectors in E_n, the vertices of the regular n-dimensional simplex D_n; y_j, $j = 0, 1, \dots, n$, are the "barycentric" coordinates (b.c.) of the vector $Y = [y_j] = \sum_j y_j e_j$ in E_n; $[y_j] = [y_j + y]$, y arbitrary;

$$y_{\text{cp}} = \frac{1}{n+1} \sum y_j, \qquad y'_j = y_j - y_{\text{cp}}$$

are the canonical b.c. of the vector $[y_j] = [y'_j]$; Ω_n is the lattice in E_n, the set of vectors $K = [k_j]$ with integral b.c. k_j; W_n is the Dirichlet domain of Ω_n, its volume $|W_n| = (n+1)^{(n-1)/2} n^{-n/2}$;

$$\sum_j = \sum_{j=0}^{n}; \qquad \prod_j = \prod_{j=0}^{\infty}; \qquad \sum_m = \sum_{m=-\infty}^{\infty}; \qquad \frac{1}{m!} = 0 \quad \text{for } m < 0.$$

Let $K = [k_j] \subset \Omega_n$

$$U_K^{(n)}(x) = U_{(k_j)}^{(n)}(x) = U_{(k_0, k_1, \dots, k_n)}^{(n)}(x) = \sum_m \prod_j \frac{x^{m+k_j}}{(m+k_j)!}. \tag{1}$$

For $Z = [z_j] = [z'_j]$, $Y = [y_j] = [y'_j]$,

$$\langle Z, Y \rangle = \sum_j z_j y'_j = \sum_j y_j z'_j.$$

1. Let there be given m indices i (for example, $i = 0, 1, \dots, m-1$ or $i = 1, 2, \dots, m$) and m numbers a_i with indices i; let s be a permutation of these indices, $i \to si$. Then $sa_i = a_{si}$; S_m is the group of the $m!$ permutations s. If, among the numbers a_i, l_p are equal to p, $\sum_p l_p = m$, then $a(a_i) = \prod_p l_p!$ is the number of permutations $s \subset S_m$ for which all $sa_i = a_i$. We shall, in particular, consider the group S_{n+1} of permutations s of the $(n+1)$

indices $j = 0, 1, \dots, n$ of the b.c. of the vector $Y = [y_j]$; we denote $sY = [sy_j] = [y_{sj}]$; thus s may be regarded as a transformation of E_n: $Y \to sY$, and the group S_{n+1} as the group of transformations of the "simplex D_n" of the space E_n. Each vector $K = [k_j] \subset \Omega_n$ passes under transformations $s \subset S_{n+1}$ into vectors sK, differing only in the order of the b.c.; the number of distinct ones among them is $(n+1)!/a(k_j)$; we denote the set of these vectors by $\mathcal{K} = \{k_j\}$; $a(\mathcal{K}) = a(k_j)$, and the set of all $\mathcal{K}$' s by $\widetilde{\Omega}_n$.

2. The functions $U_K^{(n)}(x) = U_{(k_1 \dots k_n)}(x)$ are symmetric in the k_j. We shall denote by $U_K^{(n)}(x)$ all functions $U_K^{(n)}(x)$ that are equal to one another, $K \subset \mathcal{K}$.

Let $\prod_j t_j = 1$, $K = [k_j]$, $t^K = \prod_j t_j^{k_j}$; for $K \subset \mathcal{K} = \{k_j\}$,

$$A_n(\mathcal{K}, t) = A_n(t_j, k_j) = \sum_{s \subset S_{n+1}} t^{sK}. \tag{2}$$

The generating formula for $U_K^{(n)}(x)$ from (1) takes the form

$$\exp\left(x \sum_j t_j\right) = \sum_{K \subset \widetilde{\Omega}_n} \frac{1}{a(\mathcal{K})} A_n(\mathcal{K}, t)\, U_K^{(n)}(x). \tag{3}$$

For $t_j = e^{y'_j}$ $\left(\sum_j y'_j = 0,\ [y'_j] = Y'\right)$

$$B_n(\mathcal{K}, Y') = B_n(k_j, y'_j) = A_n(k_j, t_j) = \sum_{s \subset S_{n+1}} e^{\langle sK, Y \rangle}, \tag{4}$$

$$C_n(\mathcal{K}, Y) = B_n(\mathcal{K}, 2\pi i Y). \tag{5}$$

Replacing t_j in (3) by $e^{y'_j}$ $(e^{2\pi i y_j})$ leads to replacing $A_n(\mathcal{K}, t)$ by $B_n(\mathcal{K}, Y)$ $(C_n(\mathcal{K}, Y))$. The integral representation from (1) for $U_K^{(n)}(x)$ takes the form

$$U_K^{(n)}(x) = \frac{1}{|d_n|} \int_{d_n} \cdots \int \exp\left(\sum_j x e^{2\pi i y_j}\right) \overline{C_n(\mathcal{K}, Y)}\, d\omega_n; \tag{6}$$

d_n is the set of points W_n for which $y_0 \le y_1 \le \dots \le y_n$; $|d_n| = \dfrac{1}{(n+1)!}|W_n|$. Note that the system $C_n(\mathcal{K}, Y)$, $\mathcal{K} \subset \Omega_n$, as functions of Y, forms a complete orthogonal system on d_n, and every function $f(Y)$ with integrable square on d_n is expanded there in the Fourier series

$$f(Y) = \sum_{K \subset \Omega_n} \frac{C_K}{a(\mathcal{K})} C_n(\mathcal{K}, Y), \qquad C_K = \frac{1}{|d_n|} \int_{d_n} \cdots \int f(Y) \overline{C_n(\mathcal{K}, Y)}\, d\omega_n. \quad (7)$$

Remark. Every function $f(Y)$ defined on d_n is extended by the equalities $f(sY) = f(Y)$, $s \subset S_{n+1}$, $f(Y + e_j) = f(Y)$ to a function symmetric with respect to the b.c. in E_n with vector periods e_j (such are $C_n(\mathcal{K}, Y)$).

3. We consider o.f.—homogeneous forms and s.f.—symmetric o.f. in E_n; when passing from d.c. to b.c. and conversely, the homogeneity and degree of an o.f. do not change. We distinguish s.f.d. and s.f.b.—s.f. with respect to d.c. and b.c. in E_n. Sph.f.—spherical functions—are o.f. satisfying Laplace' s equation (simultaneously with respect to d.c. and b.c.); s.sph.d. (and s.sph.b.) are simultaneously sph.f. and s.f.d. (s.f.b.). [o.f.$_{p,n}$], [s.f.b.$_{p,n}$], etc. are linear systems of all o.f. (s.f.b., etc.) in E_n of degree p; their dimensions are dim[o.f.$_{p,n}$], etc. In what follows p, n, m, l, p_i are natural numbers.

We consider representations of p by sums

$$p = \sum_{i=1}^{l} p_i$$

under restrictions on p_i and l: a) $p_i \geq m$; b) $p_i \leq n$; c) $p_i \neq k$; d) $l \leq n$.

We denote the number of such representations of p under the conditions: a) by $\alpha(p, m)$; a) and b) by $\alpha(p, m, n)$; a), b), and c) by $\alpha(p, m, n, k)$; a) and d) by $\beta(p, m, n)$.

Obviously, for $n \geq p$, $\alpha(p, m) = \alpha(p, m, n) = \beta(p, m, n)$. Further, $\beta(p, 1, n) = \alpha(p, 1, n)$, $\beta(p, m, n) > \alpha(p, m, n)$ for $m > 1,\ n < p$; $\alpha(p, m, n, k) = \alpha(p, m, n) - \alpha(p - k, m, n)$.

As $p \to \infty$, $m < n$,

$$\alpha(p, m, n) = \frac{(m-1)!}{n!(n-m)!} p^{n-m} + O(p^{n-m-1}).$$

$$\dim[\text{s.f.d.}_{p,n}] = \alpha(p, 1, n); \qquad \dim[\text{s.f.b.}_{p,n}] = \alpha(p, 2, n+1);$$

$$\dim[\text{s.sph.d.}_{p,n}] = \alpha(p, 1, n, 2); \qquad \dim[\text{s.sph.b.}_{p,n}] = \alpha(p, 3, n+1),$$

$$\lim_{p\to\infty} \frac{\dim[\text{s.f.d.}_{p,n}]}{\dim[\text{o.f.}_{p,n}]} = \lim_{p\to\infty} \frac{\dim[\text{s.sph.d.}_{p,n}]}{\dim[\text{sph.f.}_{p,n}]} = \frac{1}{n!},$$

$$\lim_{p\to\infty}\frac{\dim[\text{s.f.b.}_{p,n}]}{\dim[\text{o.f.}_{p,n}]}=\lim_{p\to\infty}\frac{\dim[\text{s.sph.b.}_{p,n}]}{\dim[\text{sph.f.}_{p,n}]}=\frac{1}{(n+1)!}.$$

All symmetric functions of m variables $z_1, z_2, \dots, z_m$ of degree p are linear combinations of the simplest such symmetric functions $T^{(m)}_{(r_i)} = T^{(m)}_{r_1,r_2,\dots,r_m}$, "generated" by the products $\prod_i z_i^{r_i}$:

$$T^{(m)}_{(r_i)}(z_i)=\frac{1}{a(r_i)}\sum_{s\subset S_m}\prod_i (sz_i)^{r_i},\qquad \sum_i r_i=p,\qquad r_i\geqslant 0.$$

In [s.f. $d_{n,p}$] the forms $T^{(n)}_{(r_i)}$ in the variables x_i, $\sum_i r_i = p$, form a basis.

In [s.f. $b_{n,p}$] all forms are expressed linearly in terms of $T^{(n+1)}_{(r_j)}(y'_j)$, $\sum_j r_j = p$, where $r_j \neq 1$; for $n+1 \geqslant p$ they form a basis, while for $n+1 < p$ there exist $\beta(p,2,n+1) - \alpha(p,2,n+1) > 0$ linear relations among them. For example, in [s.f. $b_{n,5}$] for $n \geqslant 4$ the basis is formed by $T^{(n+1)}_5(y'_j) = \sum_j y^5_{j'}$ and $T^{(n+1)}_{3,2} = \sum_{i\neq j} y^3_{i'} y^2_{j'}$ (we omit the indices $r_j = 0$ in the notation $T^{(n+1)}_{(r_j)}$). If $n < 4$, $(T^{(n+1)}_5 - 5T^{(n+1)}_{3,2})(y'_j) = 0$; for $n = 1$, $T^{(n+1)}_5 = T^{(n+1)}_3 = 0$.

The symmetrized exponential functions $B_n(\mathcal{K}, Y) = B_n(k_j, y'_j)$ are expanded in a series in the s.f.b. $\left([T]^2 = T(k'_j)T(y'_j)\right)$

$$B^{(n)}(\mathcal{K},Y)=(n+1)!+\sum_{p=2}^{\infty}a_{p,n}(k_j,y'_j),$$

$$a_{p,n}(k_j,y'_j)=\sum_{r_j\geqslant 0}\cdots\sum_{\sum_j r_j=p}\frac{a(r_j)\,p!}{\prod_j r_j!}\,[T^{(n+1)}_{(r_j)}]^2. \tag{8}$$

Expressing all $T^{(n+1)}_{(k_j)}(y'_j)$ in terms of the basis functions, we obtain for $p = 2,3,4,5$ (when $n+1 < p$ the terms containing the factor $(n-k)!$, where $k > n$, vanish):

$$a_{2,n}(k_j,y'_j)=(n+1)(n-1)![T^{(n+1)}_2]^2,$$

$$a_{3,n}(k_j,y'_j)=(n+1)^2(n-2)![T^{(n+1)}_3]^2,$$

$$\begin{aligned}a_{4,n}(k_j,y'_j)&=(n+4)(n-1)![T^{(n+1)}_4]^2+12(n-1)![T^{(n+1)}_{2,2}]^2\\&\quad+(6n-3)(n-3)![T^{(n+1)}_4-2T^{(n+1)}_{2,2}]^2,\end{aligned}$$

$$a_{5,n}(k_j, y'_j) = (n-2)!\{(n+5)(n-1)[T_5^{(n+1)}]^2 + 10(n+5)[T_{3,2}^{(n+1)}]^2 + 10[T_5^{(n+1)} - T_{3,2}^{(n+1)}]^2\} + (10n-14)(n-4)![T_5^{(n+1)} - 5T_{3,2}^{(n+1)}]^2.$$

Moscow State University
named after M. V. Lomonosov

Received
12 X 1967

CITED LITERATURE

1. L. A. Lyusternik, *DAN*, **177**, No. 5 (1967).

Note: Figure translations are in progress. See original paper for figures.