DIFFERENCE OPERATORS IN CONVOLUTIONS

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Abstract

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MATHEMATICS

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DIFFERENCE OPERATORS IN CONVOLUTIONS

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0. Introduction. An extremely convenient tool for the study and numerical solution of problems for differential, as well as more general pseudodifferential (see, for example, ([1])) operators is provided by the difference schemes that approximate them. The difference operators arising in this connection may also be considered independently of the problems that they approximate. In the present note an algebra of such operators is constructed, and from them a class of elliptic difference schemes is singled out, analogous in its properties to elliptic pseudodifferential operators. Concrete difference approximations of strongly elliptic equations of high order have also been considered earlier (see, for example, ([2–4])), but the general concept of ellipticity has hitherto been absent.

1. Spaces of grid functions H_s^*. On the uniform grid R_h^n with mesh h in the Euclidean space R^n we consider grid functions with values in the Euclidean space R^p, depending on the parameter h, $0 < h \le h_0$.

By definition, a grid function $u(h,x) \in H_s^* = H_s^*(R_h^n)$, if

$$\sup_{0<h\le h_0} \|u\|_s^* = \left(\int_{|h\xi_j|\le\pi} \left(1 + \sum_{k=1}^{n} \frac{4\sin^2 h\xi_k/2}{h^2} \right)^s \left| F_{x\to\xi} u \right|^2 d\xi_1 \dots d\xi_n \right)^{1/2} < \infty. \tag{1}$$

Here $F_{x\to\xi}u$ is the grid Fourier transform of the function $u(h,x)$ (the reconstruction of a periodic function from its Fourier coefficients). For integer $s \ge 0$ the norm (1) is equivalent to the following:

$$\|u\|_s^* \sim \left(\sum_{x\in R_h^n} \sum_{|\alpha|\le s} |D^\alpha u|^2 h^n \right)^{1/2},$$

where $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$, D_j is the operator of difference differentiation forward with respect to x_j on the grid R_h^n, multiplied by $-i$ (the operator of difference differentiation backward with respect to x_j, multiplied by $-i$, will be denoted by $\overline{D}_j$); $|\alpha| = \alpha_1 + ... + \alpha_n$, and the α_j are nonnegative integers. We formulate two assertions concerning the connection between the spaces H_s^* and the spaces $H_s(R^n) = W_{2,s}(R^n)$.

Proposition 1. Let $u(x) \in H_s(R^n)$. Then for every $s' \geq 0$ there exists a function $v(h,x) \in H_s(R^n)$, for all h, $0 < h \leq h_0$, such that $\|u - v\|_{s-s'} \leq C_1 h^{s'} \|u\|_s$, the restriction of $v(h,x)$ to R_h^n belongs to H_s^*, and $\|v\|_s^* \leq C_2 \|u\|_s$. Here C_1 and C_2 depend only on s, s', and h_0.

Remark. The restriction of a function $u(x) \in H_s(R^n)$ to R_h^n, generally speaking, does not belong to $H_s^*(R_h^n)$, but belongs to $H_t^*(R_h^n)$ for $t < s - n/2$.

Proposition 2. Let $u(h,x) \in H_s^*(R_h^n)$ and $s > n/2$. Then for each h, $0 < h \leq h_0$, $u(h,x)$ can be extended to all of R^n so that the extension $l_{hu} = v(h,x) \in H_s(R^n)$ and $\|v\|_s \leq C\|u\|_s^*$, where C is independent of h.

Put

$$H_\infty^* = \bigcap_{-\infty < s < \infty} H_s^*.$$

We also introduce into consideration the grid analogue S^* of the Schwartz space S. By definition, $u \in S^*$ if all its -

derivatives are uniformly bounded with respect to h and tend to zero as $|x| \to \infty$ faster than any power of $|x|^{-1}$.

2. Definition of difference operators in convolutions (d.o.c.). Let C_ζ^n be the complex n-dimensional space, and let R_+^1 be the half-axis $\{h > 0\}$. Denote by T_n the manifold lying in $C_\zeta^n \times R_+^1$, defined by the relations $|1 + ih\zeta_k| = 1$, $1 \leq k \leq n$, where $h \in R_+^1$, $\zeta = (\zeta_1, ..., \zeta_n) \in C_\zeta^n$. The section of T_n by the surface $h = h_1$ is the torus $T_{n,h_1} = \{\zeta : |1 + ih_1\zeta_k| = 1, 1 \leq k \leq n\}$. Let $a(x; h, \zeta, \bar\zeta)$ be a matrix function of size $p \times p$, defined on $R_h^n \times T^n$. We shall assume that: 1) $a(x; h, \zeta, \bar\zeta)$ is continuous on T^n for $\zeta \neq 0$; 2) $a(x; h, \zeta, \bar\zeta)$ is infinitely differentiable with respect to $(\zeta, \bar\zeta)$ for $\zeta \in T_{n,h}$, $h > 0$, $\zeta \neq 0$; 3) $a(x; h, \zeta, \bar\zeta)$ is a positively homogeneous function of $(h^{-1}, \zeta, \bar\zeta)$ of order γ, $\gamma \in R^1$, i.e. $a(x; t^{-1}h, t\zeta, t\bar\zeta) = t^\gamma a(x; h, \zeta, \bar\zeta)$ for all $t > 0$ and some real γ; 4) $|a(x; 1, \zeta, \bar\zeta)| \leq c|\zeta|^\gamma$ for $\zeta \in C^n$, $|1 + i\zeta_k| = 1$, $1 \leq k \leq n$, $\zeta \neq 0$, where $|a|$ is the norm of the matrix a in R^p; 5) $a(x; h, \zeta, \bar\zeta) \equiv a(\infty; h, \zeta, \bar\zeta)$ for $|x| \geq R$, $(h, \zeta) \in T^n$, $\zeta \neq 0$ (R is a constant), and $a'(x; h, \zeta, \bar\zeta) = a(x; h, \zeta, \bar\zeta) - a(\infty; h, \zeta, \bar\zeta)$ belongs, in the first argument, to S^*.

Definition 1. A homogeneous d.o.c. with symbol $a(x; h, \zeta, \bar\zeta)$ of order γ is defined on functions $u \in S^*$ by the equality

$$Au = F_{\zeta \to x}^{-1} a(x; h, \zeta, \bar\zeta) F_{x \to \zeta} u. \tag{2}$$

Here $F_{x\to\zeta}$ and $F^{-1}_{\zeta\to x}$ are, respectively, the direct and inverse mesh Fourier transforms, $x \in R^n_h$, $(h,\zeta) \in T^n$, $\zeta_k = (e^{ih\xi_k} - 1)/ih,\ 1 \le k \le n$.

For $\gamma < 0$, in order to give meaning to formula (2), before applying the operator $F^{-1}_{\zeta\to x}$ one should multiply the symbol of the operator A by a cutoff infinitely differentiable function which is equal to zero in a neighborhood of the point $\zeta = 0$ and equal to one outside some larger neighborhood of this point.

Theorem 1. Let A be a homogeneous d.o.c. of degree of homogeneity γ. Then A is a bounded operator from H^*_s to $H^*_{s-\gamma}$, and the norm of A is uniformly bounded with respect to $h,\ 0 < h \le h_0$.

The proof of Theorem 1 is based on the equivalent representation of a homogeneous d.o.c. in the form

$$\widetilde{A}u = F_{x\to\xi}(Au) = a(\infty, h, \zeta, \bar\zeta)\tilde u + (2\pi)^{-n/2} \int_{|h\eta_j|\le\pi} \tilde a'(\xi-\eta; h, \omega, \bar\omega)\tilde u(h,\eta)\, d\eta, \tag{3}$$

where $a'(x; h, \zeta, \bar\zeta) = a(x; h, \zeta, \bar\zeta) - a(\infty; h, \zeta, \bar\zeta)$, $\zeta = (\zeta_1, \dots, \zeta_n)$, $\omega = (\omega_1, \dots, \omega_n)$, $\zeta_k = (e^{ih\xi_k} - 1)/ih$, $\omega_k = (e^{ih\eta_k} - 1)/ih,\ 1 \le k \le n$, and $\tilde a'$ is the mesh Fourier transform of the function a' with respect to the first argument.

3. Composition of two homogeneous d.o.c. The composition of two homogeneous d.o.c., generally speaking, is no longer a homogeneous d.o.c. However, the following assertion is true.

Theorem 2. Let A be a homogeneous d.o.c. of order γ with symbol $a(x; h, \zeta, \bar\zeta)$, and let B be a homogeneous d.o.c. of order λ with symbol $b(x; h, \zeta, \bar\zeta)$. Then the composition AB can be decomposed into the sum, with arbitrary integer $\rho > 0$,

$$AB = \sum_{j=0}^{\rho-1} C_j + T_\rho, \tag{4}$$

where C_j are homogeneous d.o.c. of orders $\gamma + \lambda - j$, and the operator T_ρ is a continuous operator from H^*_s to $H^*_{s-(\gamma+\lambda-\rho)}$ for any s. The symbols of the operators C_j are determined by the equality

$$c_j(x; h, \zeta, \bar\zeta) =$$

$$= \sum_{|\alpha|+|\beta|=j} \frac{1}{\alpha!\beta!} (1 + ih\zeta)^\alpha (1 - ih\bar\zeta)^\beta \partial^\alpha \bar\partial^\beta a(x; h, \zeta, \bar\zeta)\, D^\alpha \bar D^\beta b(x; h, \zeta, \bar\zeta), \tag{5}$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$, $\bar{\partial}^\beta = \bar{\partial}_1^{\beta_1} \cdots \bar{\partial}_n^{\beta_n}$, ∂_j is differentiation with respect to ζ_j, and $\bar{\partial}_j$ with respect to $\bar{\zeta}_j$, $D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, $\bar{D}^\beta = \bar{D}_1^{\beta_1} \cdots \bar{D}_n^{\beta_n}$ (D_j and $\bar{D}_j$ are defined in 1).

4. The adjoint operator. Let A be a homogeneous d.o.c. Introduce the scalar product of mesh functions

$$(u, v) = \sum_{x \in R_h^n} \bar{u} \cdot v h^n \tag{6}$$

and define the operator A^* adjoint to A with respect to the scalar product (6). A^* will no longer be a homogeneous d.o.c. However, the following holds.

Theorem 3. Let A be a homogeneous d.o.c. of order γ with symbol $a(x; h, \xi, \bar{\xi})$, and let the operator A^* be adjoint to A with respect to the scalar product (6). Then for any integer $\rho > 0$ the equality

$$A^* = \sum_{j=0}^{\rho-1} B_j + T_\rho \tag{7}$$

holds, where B_j are homogeneous d.o.c. of orders $\gamma - j$ with symbols

$$b_j(x; h, \xi, \bar{\xi}) = \sum_{|\alpha|+|\beta|=j} \frac{1}{\alpha! \beta!} (1 + ih\xi)^\alpha (1 - ih\bar{\xi})^\beta D^\alpha \bar{D}^\beta \partial^\alpha \bar{\partial}^\beta a^*(x; h, \xi, \bar{\xi}),$$

a^* is the matrix adjoint to a, and the operator T_ρ acts continuously from H_s^* to $H_{s-(\gamma-\rho)}^*$ for any s.

5. General d.o.c. Algebra of d.o.c. Up to now we have considered homogeneous d.o.c. But the operations of multiplication and taking adjoints take us out of this class of operators. Therefore, in this section a definition of general d.o.c. will be given.

Definition 2. Let $r_0, r_1, \ldots$ be a strictly monotonically decreasing sequence of real numbers ($r_k \downarrow -\infty$, if the number of terms of the sequence is infinite). Let $a_k(x; h, \xi, \bar{\xi})$ be the canonical matrix function of homogeneity r_k defined above, and let A_k be the homogeneous d.o.c. of order r_k corresponding to it. Let A be an operator on functions from S^* having the property that for any natural N the difference

$$A - \sum_{k=0}^{N} A_k$$

acts continuously from H^*_s to $H^*_{s-r_N+\varepsilon}$ for some $\varepsilon > 0$ and any s. Then the operator A is called a d.o.c. with asymptotic expansion

$$A \sim \sum_{k=0}^{\infty} A_k.$$

The full symbol $\sigma(A)$ of the operator A is the asymptotic series

$$\sigma(A) \sim \sum_{k=0}^{\infty} a_k(x; h, \xi, \bar{\xi}).$$

For d.o.c. approximating differential operators, the full symbol, as a rule, contains only a finite number of terms.

Theorem 4. Let $\{r_k\}_{k=1}^{\infty} \subset R^1$, $r_k \downarrow -\infty$ $(k \to \infty)$, and let $a_k(x; h, \xi, \bar{\xi})$ be a sequence of homogeneous symbols of orders r_k. Then there exists a d.o.c. with full symbol

$$\sum_{k=0}^{\infty} a_k(x; h, \xi, \bar{\xi}).$$

General d.o.c. are determined by their full symbol up to operators T_∞ of order $-\infty$, mapping any H^*_s into H^*_∞. It is not difficult to verify that the composition of two d.o.c. A and B, as well as the operator A^*, are again d.o.c. with full symbols

$$\sigma(AB) \sim \sum_{|\alpha|+|\beta|\geq 0} \frac{1}{\alpha!\beta!}(1 + ih\xi)^\alpha (1 - ih\bar{\xi})^\beta \partial^\alpha \bar{\partial}^\beta \sigma(A)\, D^\alpha \bar{D}^\beta \sigma(B), \tag{8}$$

$$c(A^*) \sim \sum_{|\alpha|+|\beta|\geq 0} \frac{1}{\alpha!\beta!}(1 + ih\xi)^\alpha (1 - ih\bar{\xi})^\beta \bar{\partial}_\alpha \bar{\partial}^\beta D^\alpha \bar{D}^\beta \sigma^*(A). \tag{9}$$

Thus, the general d.o.c. form an algebra with involution.

6. **Elliptic** d.o.c. Among all d.o.c. there is a distinguished class, in many respects analogous in its properties to elliptic pseudodifferential operators. It is therefore natural to call these d.o.c. elliptic.

Definition 3. A d.o.c. A of order r_0 with complete symbol

$$\sum_{k=0}^{\infty} a_k(x; h, \xi, \bar{\xi})$$

is called elliptic (briefly, e.d.o.c.) if

$$\left|a_0^{-1}(x,1,\xi,\bar{\xi})\right| \leq c|\xi|^{-r_0} \tag{10}$$

for $\xi \in C_\zeta^n$, $|1+i\xi_k|=1,\ 1\leq k\leq n,\ \xi\neq 0$, where c is a constant, and $\left|a_0^{-1}\right|$ is the norm of the matrix a_0^{-1} in R^p. The simplest example of an e.d.o.c. is the operator with symbol $-|\xi|^2$, which is the classical cross approximation of the Laplace operator. Let

$$\sum_{|\alpha|\leq m,\ |\beta|\leq m} a_{\alpha,\beta} D^{\alpha+\beta}$$

be a strongly elliptic differential operator of order $2m$. Then, as is not hard to verify, the d.o.c. with complete symbol

$$\sum_{|\alpha|\leq m,\ |\beta|\leq m} a_{\alpha,\beta} \xi^\alpha \bar{\xi}^\beta$$

will be elliptic.

Theorem 5. *If A is an e.d.o.c., then there exists an e.d.o.c. B such that $AB-I$ and $BA-I$ are operators of order $-\infty$ (here I is the identity operator).*

In proving this theorem, the complete symbol of the operator B is sought in the form of an asymptotic series

$$\sum_{k=0}^{\infty} b_k(x;h,\xi,\bar{\xi});$$

at the same time, for the matrices $b_k(x;h,\xi,\bar{\xi})$ one obtains an infinite system of recurrence relations, from which, by virtue of the ellipticity condition (10), the matrices b_k are found successively, one after another. From the complete symbol, the operator B is reconstructed up to an operator of order $-\infty$ (see Theorem 4).

Theorem 6. *Let A be an e.d.o.c. of order γ, acting from H_s^* into $H_{s-\gamma}^*$. Then, with a constant c independent of h, for $0<h\leq h_0$ the estimate*

$$\|u\|_s^* \leq c(\|Au\|_{s-\gamma}^* + \|u\|_{s'}^*) \tag{11}$$

holds, where s' is any real number, $s'<s$.

The converse assertion is also true.

Theorem 7. *If, for some s and s', $s>s'$, the estimate (11) with a constant c independent of h, $0<h\leq h_0$, holds for a d.o.c. A, then A is an e.d.o.c.*

Let us note that an elliptic differential operator may be approximated by a nonelliptic difference operator. For example, the difference approximation of the Laplace operator with complete symbol

$$-\xi_1^2 - \sum_{k=2}^{n} |\xi_k|^2$$

will not be elliptic, whereas the usual cross approximation defines an e.d.o.c.

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Note: Figure translations are in progress. See original paper for figures.