ON THE CHOICE OF ITERATION PARAMETERS IN THE METHOD OF VARIABLE DIRECTIONS FOR A DIFFERENCE DIRICHLET PROBLEM OF HIGHER ORDER OF ACCURACY

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Abstract

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ON THE CHOICE OF ITERATION PARAMETERS IN THE METHOD OF VARIABLE DIRECTIONS FOR A DIFFERENCE DIRICHLET PROBLEM OF HIGHER ORDER OF ACCURACY

A difference Dirichlet problem of higher order of accuracy $O(|h|^4)$ or $O(h^6)$, for $h_1 = h_2 = h$, is considered for the Poisson equation in a rectangle. For its solution an iterative method of variable directions is proposed with two sets of parameters $\{\tau_s\}$ and $\{\omega_s\}$. The minimax problem of choosing the optimal parameters is reduced to a problem solved by Jordan ([1]). With such a choice of parameters, the number of iterations for the scheme $O(|h|^4)$, in comparison with the scheme $O(|h|^2)$, increases only slightly (by no more than 10% for $h_1 = h_2 = h < l/10$ in the case of a square of side l).

1. Consider a scheme of higher order of accuracy on the rectangular grid $\overline{\omega}_h = \{(i_1h_1, i_2h_2),\ i_\alpha = 0, 1, \dots, N_\alpha,\ h_\alpha = l_\alpha/N_\alpha,\ \alpha = 1, 2\}$ (for notation see ([3])).

$$\Lambda' y = \left(\Lambda_1 + \Lambda_2 + \frac{1}{12}(h_1^2 + h_2^2)\Lambda_1\Lambda_2\right) y = -\varphi(x), \qquad x \in \omega_h, \qquad y|_{\gamma_h} = \mu(x), \tag{1}$$

(where $\omega_h = \{(i_1h_1, i_2h_2),\ 0 < i_\alpha < N_\alpha,\ \alpha = 1, 2\}$, γ_h is the set of boundary nodes), corresponding to the Dirichlet problem for the Poisson equation in the rectangle
$\overline{G} = (0 \le x_1 \le l_1,\ 0 \le x_2 \le l_2)$:

$$\Delta u = -f(x), \qquad x = (x_1, x_2), \qquad 0 < x_\alpha < l_\alpha, \quad \alpha = 1, 2; \qquad u|_\Gamma = \mu(x). \tag{2}$$

Here $\Lambda_\alpha y = y_{\bar{x}_\alpha x_\alpha}$, $\varphi = f + \frac{1}{12}h_1^2\Lambda_1 f + \frac{1}{12}h_2^2\Lambda_2 f$, and Γ is the boundary of the rectangle. Scheme (1) has accuracy $O(|h|^4)$, $|h|^2 = h_1^2 + h_2^2$, and for $h_1 = h_2 = h$

and the corresponding φ, accuracy $O(h^6)$. For its solution in ([2–4]), variable-direction schemes with a cyclic set of parameters $\{\tau_s\}$ were proposed. However, as follows from ([1]), a cyclic set of parameters is not optimal even in the case of schemes $O(|h|^2)$.

2. For solving problem (1), the following variable-direction scheme with two sets of parameters $\{\tau_s\}$ and $\{\omega_s\}$ is proposed:

$$(E-(\tau_s-\varkappa_1)\Lambda_1)\,y^{s+1/2}=(E+(\tau_s+\varkappa_2)\Lambda_2)\,y^s+(\tau_s-\varkappa_1)\varphi;$$

$$y^{s+1/2}=\overline{\mu}\quad\text{for } x_1=0,l_1;\quad 0<x_2<l_2;\qquad \overline{\mu}=\mu+(\varkappa_1+\varkappa_2)\Lambda_2\mu;$$

$$(E-(\omega_s-\varkappa_2)\Lambda_2)\,y^{s+1}=(E+(\omega_s+\varkappa_1)\Lambda_1)\,y^{s+1/2}+(\omega_s+\varkappa_1)\varphi;\qquad (3)$$

$$y^{s+1}=\mu(x)\quad\text{for } x_2=0,l_2,$$

where $\varkappa_\alpha=\frac{1}{12}h_\alpha^2,\ \alpha=1,2$, and E is the identity operator ($Ey=y$).

If equation (1) is written in matrix form, then the boundary conditions on γ_h will be homogeneous, $y|_{\gamma_h}=0$, and the right-hand side φ in (3) is replaced by a function $\widetilde{\varphi}$, which differs from φ only at the near-boundary nodes:

$$\widetilde{\varphi}=\varphi\quad\text{for } h_\alpha<x_\alpha<l_\alpha-h_\alpha,\quad \alpha=1,2;$$

$$\widetilde{\varphi}=\varphi+\frac{1}{h_\alpha^2}(\mu^{-1\alpha}+(\varkappa_1+\varkappa_2)\Lambda_\beta\mu)\quad\text{for } x_\alpha=h_\alpha,\quad \alpha\neq\beta,\quad \alpha,\beta=1,2;$$

$$\widetilde{\varphi}=\varphi+\frac{1}{h_\alpha^2}(\mu^{+1\alpha}+(\varkappa_1+\varkappa_2)\Lambda_\beta\mu)\quad\text{for } x_\alpha=l_\alpha-h_\alpha,\quad \alpha\neq\beta,\quad \alpha,\beta=1,2.$$

Then homogeneous boundary conditions will be imposed:

$$y^s=y^{s+1}=y^{s+1/2}=0\quad\text{for } x\in\gamma_h,$$

and in equations (3) one must write $\tilde{\varphi}$ instead of φ.

The optimal set of parameters $\{\tau_s\}$ and $\{\omega_s\}$ is found by Jordan' s method (1) for the operators $-\Lambda_\alpha$ ($\alpha=1,2$), whose smallest and largest eigenvalues are determined by the formulas

$$\tilde{a}_\alpha = \frac{a_\alpha}{1 - h_\alpha \chi_\alpha}, \qquad \tilde{b}_\alpha = \frac{b_\alpha}{1 - b_\alpha \chi_\alpha}, \qquad \text{where } a_\alpha = \frac{4}{h_\alpha^2} \sin^2 \frac{\pi h_\alpha}{2 l_\alpha}, \quad b_\alpha = \frac{4}{h_\alpha^2} \cos^2 \frac{2\pi h_\alpha}{2 l_\alpha}. \tag{4}$$

3. Let us pass to the justification of the proposed method. Consider the operator equation

$$A' v = \varphi, \qquad A' = A_1 + A_2 - (\varkappa_1 + \varkappa_2) A_1 A_2, \qquad \varkappa_1 > 0, \ \varkappa_2 > 0, \tag{5}$$

where A_1 and A_2 are linear operators defined on a finite-dimensional linear space H with scalar product $(\ ,\ )$, and $\varphi \in H$.

Assume that:

I. A_1 and A_2 are positive self-adjoint operators with bounds a_1, b_1 and a_2, b_2, respectively, so that $a_\alpha E \le A_\alpha \le b_\alpha E$ $(a_\alpha > 0)$, $\alpha = 1, 2$, or $a_\alpha (x, x) \le (A_\alpha x, x) \le b_\alpha (x, x)$ for all $x \in H$.

II. $\varkappa_\alpha < \dfrac{1}{b_\alpha}$, so that the positive definite operators $(E - \varkappa_\alpha A_\alpha)^{-1}$, $\alpha = 1, 2$, exist.

III. The operators A_1 and A_2 are permutable, $A_1 A_2 = A_2 A_1$.

Lemma 1. *If conditions I and II are satisfied, then the operator*

$$\widetilde{A}_\alpha = (E - \varkappa_\alpha A_\alpha)^{-1} A_\alpha \qquad (\alpha = 1, 2) \tag{6}$$

has bounds $\tilde{a}_\alpha$ and $\tilde{b}_\alpha$, determined by the formulas

$$\tilde{a}_\alpha = \frac{a_\alpha}{1 - \varkappa_\alpha a_\alpha}, \qquad \tilde{b}_\alpha = \frac{b_\alpha}{1 - \varkappa_\alpha b_\alpha}, \qquad \alpha = 1, 2. \tag{7}$$

Represent $\widetilde{A}_\alpha$ in the form $\widetilde{A}_\alpha = (A_\alpha^{-1} - \varkappa_\alpha E)^{-1}$. From the condition $a_\alpha E \le A_\alpha \le b_\alpha E$, in view of the self-adjointness of A_α, A_α^{-1}, and $A_\alpha^{-1} - \varkappa_\alpha E > 0$, it follows that

$$\frac{1}{b_\alpha} E \le A_\alpha^{-1} \le \frac{1}{a_\alpha} E, \qquad \left(\frac{1}{b_\alpha} - \varkappa_\alpha \right) E \le A_\alpha^{-1} - \varkappa_\alpha E \le \left(\frac{1}{a_\alpha} - \varkappa_\alpha \right) E;$$

since

$$\varkappa_\alpha < \frac{1}{b_\alpha},$$

we have $\tilde{a}_\alpha E \le \widetilde{A}_\alpha = (A_\alpha^{-1} - \varkappa_\alpha E)^{-1} \le \tilde{b}_\alpha E$. The lemma is proved.

We note that the lemma is valid for nonpermutable operators A_1 and A_2 defined in a Hilbert space of any number of dimensions.

Theorem. *If conditions I-III are satisfied, then equation (5) is equivalent to the equation*

$$(\widetilde{A}_1 + \widetilde{A}_2)v = \tilde{\varphi}, \qquad \tilde{\varphi} = (E - \varkappa_1 A_1)^{-1}(E - \varkappa_2 A_2)^{-1}\varphi, \tag{8}$$

where $\widetilde{A}_1$ and $\widetilde{A}_2$ are determined according to (6) and are self-adjoint positive operators with bounds $\tilde{a}_1, \tilde{b}_1$ and $\tilde{a}_2, \tilde{b}_2$.

Indeed, (5) can be rewritten in the form

$$A_1(E - \varkappa_2 A_2)v + (E - \varkappa_1 A_1)A_2 v = \varphi. \tag{8'}$$

Applying to (8′) the operator $(E-\varkappa_1 A_1)^{-1}(E-\varkappa_2 A_2)^{-1}$ and taking into account the permutability of all the operators, we obtain (8). The reverse course of reasoning is obvious.

Thus, the solution of equation (5) has been reduced to the solution of equation (8) with permutable, self-adjoint, and positive operators.

$\widetilde{A}_1$ and $\widetilde{A}_2$; the eigenvalues of the operators $\widetilde{A}_1$ and $\widetilde{A}_2$ belong to the intervals $[\tilde{a}_1, \tilde{b}_1]$, $[\tilde{a}_2, \tilde{b}_2]$.

The alternating-direction iterative scheme for solving equation (8) was considered in [1], where, in particular, a method due to Jordan for choosing optimal (more precisely, practically optimal) iteration parameters $\{\tau_s\}$, $\{\omega_s\}$ is given. Let us write the corresponding scheme [1]

$$(E + \tau_s \widetilde{A}_1)(E + \omega_s \widetilde{A}_2)y^{s+1} = (E - \omega_s \widetilde{A}_1)(E - \tau_s \widetilde{A}_2)y^s + (\tau_s + \omega_s)\tilde{\varphi}.$$

Apply to both sides of this equation the operator $(E - \varkappa_1 A_1)(E - \varkappa_2 A_2)$ and take into account that all the operators are permutable and

$$(E - \varkappa_1 A_1)(E + \tau_s A_1) = E - \varkappa_1 A_1 + \tau_s(E - \varkappa_1 A_1)\widetilde{A}_1 = E + (\tau_s - \varkappa_1)A_1,$$

$$(E - \varkappa_1 A_1) \times (E - \omega_s A_1) = E - (\omega_s - \varkappa_1)A_1$$

and so on. As a result we obtain the scheme

$$\begin{aligned}(E + (\tau_s - \varkappa_1)A_1)(E + (\omega_s - \varkappa_2)A_2)y^{s+1} = \\ = (E - (\omega_s + \varkappa_1)A_1)(E - (\tau_s + \varkappa_s)A_2)y^s + (\tau_s + \omega_s)\varphi.\end{aligned} \tag{9}$$

Writing it in canonical form,

$$(E+(\tau_s-\varkappa_1)A_1)(E+(\omega_s-\varkappa_2)A_2)\frac{y^{s+1}-y^s}{\tau_s+\omega_s}+A'y^s=\varphi,$$

we see that it approximates (4) exactly on the solution v. We implement scheme (9) by means of the algorithm

$$(E+(\tau_s-\varkappa_1)A_1)y^{s+1/2}=(E-(\tau_s+\varkappa_2)A_2)y^s+(\tau_s-\varkappa_1)\varphi,$$

$$(E+(\omega_s-\varkappa_2)A_2)y^{s+1}=(E-(\omega_s+\varkappa_1)A_1)y^{s+1/2}+(\omega_s+\varkappa_1)\varphi. \tag{10}$$

The equivalence of (9) and (10) is proved by analogy with (5). From (10) we find

$$\begin{aligned}(\omega_s+\tau_s)y^{s+1/2}=(\omega_s+\varkappa_1)(E-(\tau_s+\varkappa_2)A_2)y^s+\\ +(\tau_s-\varkappa_1)(E+(\omega_s-\varkappa_2)A_2)y^{s+1}.\end{aligned} \tag{11}$$

Substituting (11) into (10), we obtain (9). The reverse course of the reasoning is obvious.

We note that all the preceding arguments remain valid also in the case of an abstract Hilbert space H, if A_1 and A_2 satisfy conditions I–III.

The results obtained in this section are, evidently, applicable not only to the Dirichlet problem (1), but also to other problems leading to an equation of the form (5).

4. Let us now turn to the scheme of increased order of accuracy (1). In this case

$$A_\alpha=-\Lambda_\alpha,\qquad \varkappa_\alpha=-\frac{1}{12}h_\alpha^2,\qquad a_\alpha=\frac{4}{h_\alpha^2}\sin^2\frac{\pi h_\alpha}{2l_\alpha},\qquad b_\alpha=\frac{4}{h_\alpha^2}\cos^2\frac{\pi h_\alpha}{2l_\alpha},\quad \alpha=1,2.$$

If, for $y^{s+1/2}$, we want to impose the ordinary conditions on γ_h, then one should use equation (11) for $x_1=0,l_1$. Indeed, putting in (11) $y^s=y^{s+1}=\mu,\ x\in\gamma_h$, we obtain

$$y^{s+1/2}=\overline{\mu},\qquad \overline{\mu}=\mu+(\varkappa_1+\varkappa_2)\Lambda_2\mu\quad \text{for } x_1=0,l_1,\ 0<x_2<l_2.$$

The order of computation is: 1) $\tilde{a}_\alpha$ and $\tilde{b}_\alpha$ are computed; 2) from $\tilde{a}_\alpha,\tilde{b}_\alpha$, according to [1], the parameters $\{\tau_s\}$ and $\{\omega_s\}$ corresponding to scheme (8) are found; 3) after

this tridiagonal-matrix algorithm solves system (3) (its solvability follows from the fact that $\tilde{A}_\alpha > 0,\ \tau_s > 0,\ \omega_s > 0$).

In [1] an approximate formula was obtained for the number of iterations $\nu(\varepsilon)$ ensuring accuracy $\varepsilon > 0$. For problem (5) it has the form

$$\tilde{\nu}(\varepsilon) \doteq \frac{1}{\pi^2} \ln \frac{4}{\tilde{\eta}} \ln \frac{4}{\varepsilon}, \tag{12}$$

where

$$\tilde{\eta} = \frac{1-\tilde{\xi}}{1+\tilde{\xi}}, \qquad \tilde{\xi} = \sqrt{\frac{(\tilde{b}_1 - \tilde{a}_1)(\tilde{b}_2 - \tilde{a}_2)}{(\tilde{b}_1 + \tilde{a}_2)(\tilde{b}_2 + \tilde{a}_1)}}.$$

Using this formula, it is not difficult to compare the numbers of iterations for the five-point scheme of second-order accuracy ($\nu(\varepsilon)$) and for the scheme of higher order of accuracy ($\tilde{\nu}(\varepsilon)$). From (12) it is clear that

$$\frac{\tilde{\nu}(\varepsilon)}{\nu(\varepsilon)} \doteq \ln \frac{4}{\tilde{\eta}} \Big/ \ln \frac{4}{\eta},$$

where η is determined by the same formulas as $\tilde{\eta}$, if in them $\tilde{a}_\alpha$ is replaced by a_α and $\tilde{b}_\alpha$ by b_α. We give the results of the comparison for the case of a square with side $l_1 = l_2 = 1$ and a square mesh $h_1 = h_2 = h$ ($\tilde{\eta} = \tilde{a}/\tilde{b},\ \eta = a/b,\ a_1 = a_2 = a,\ b_1 = b_2 = b$)

$$\frac{\tilde{\nu}}{\nu} \doteq \begin{cases} 1.1 & \text{for } h = 0.1, \\ 1.05 & \text{for } h = 0.02, \\ 1.04 & \text{for } h = 0.01. \end{cases}$$

The amount of computation per iteration for both schemes is practically the same, while the difference in the number of iterations is insignificant. Since the scheme of higher order of accuracy makes it possible to use a coarser mesh to attain a prescribed accuracy, its application is especially advantageous in those cases where the solution $u = u(x)$ of problem (2) possesses sufficient smoothness.

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Note: Figure translations are in progress. See original paper for figures.