ON THE THEORY OF THE NONLINEAR SKIN EFFECT

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Abstract

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ON THE THEORY OF THE NONLINEAR SKIN EFFECT

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The penetration of a quasistationary electromagnetic field ($H \gg E$), excited by a traveling wave of surface-current density $j_y = i_0[\delta(x-b)-\delta(x+b)]\cos(\omega t-\gamma z)$, into a plasma layer of thickness $2a$ ($a < b$) with conductivity σ and density n is considered. Since $H \gg E$, the Hall term must be taken into account in Ohm' s law, as a result of which the problem becomes nonlinear. In a reference frame moving with the phase velocity of the wave $v_\phi = \omega/\gamma$ ($v_\phi \ll c$), the induction electric field is absent, and all quantities depend only on x and z. Further setting $\mathbf{j} = \dfrac{i_0}{a}\mathbf{j}$, $\mathbf{H} = \dfrac{4\pi i_0}{c}\mathbf{H}$, $x = au$, $z = \dfrac{v}{\gamma}$, we shall have the following system of equations for the dimensionless $\mathbf{j}$ and $\mathbf{H}$ in the plasma:

$$\beta\,\partial H_y/\partial v = j_x, \quad \partial H_y/\partial u = j_z,$$

$$\beta\,\partial H_x/\partial v - \partial H_z/\partial u = j_y, \quad \partial H_x/\partial u + \beta\,\partial H_z/\partial v = 0; \tag{1}$$

$$j_y = \alpha Q_y, \quad \partial j_x/\partial u + \beta\,\partial j_z/\partial v = 0,$$

$$\beta\,\partial j_x/\partial v - \partial j_z/\partial u = \alpha(\beta\,\partial Q_x/\partial v - \partial Q_z/\partial u), \tag{2}$$

$$\mathbf{Q} = [\mathbf{e}_z \cdot \mathbf{H}] - \Omega[\mathbf{j} \cdot \mathbf{H}],$$

where $\alpha = 4\pi\sigma\omega a/\gamma c^2$, $\beta = \gamma a$, $\Omega = i_0\gamma/en\omega a$.

Outside the plasma, $\mathbf{H}$ is determined from system (1) with the prescribed density of the external current. At the boundary of the layer ($u = \pm 1$) all components of the magnetic field are continuous. From systems (1)–(2) one can eliminate $\mathbf{j}$; however, the resulting equations for $\mathbf{H}$ are complicated, and when they are solved by numerical methods additional difficulties arise, associated with the necessity of matching the fields at the boundary of the layer [1].

Let us approach the solution of this problem in another way. The rough picture of the establishment of the distributions of fields and currents in the plasma after the external current wave is switched on is as follows: magnetic and induction electric fields are excited throughout space, inducing currents in the plasma. These currents excite their own fields, and so on, until equilibrium distributions are established. This suggests trying to find the solution of the problem by analogy with the process of establishment. The process has to be divided into a number of discrete stages, assuming that at each of them the current reaches the values corresponding to Ohm' s law, although in reality the process proceeds smoothly. In solving by this method, the main difficulties associated with the nonlinearity of the problem disappear, since it is not the differential equations that become nonlinear, but the algebraic operations.

We shall seek the solution for $\mathbf{H}$, $\mathbf{j}$, $\mathbf{Q}$ in the form

$$\mathbf{H} = \sum_{k=0}^{\infty} [\mathbf{H}^{1,k}(u)\cos kv + \mathbf{H}^{2,k}(u)\sin kv] \quad \text{and so on.} \tag{3}$$

It is not difficult to verify that $H_x^{1,0} = H_z^{1,0} = j_y^{1,0} = j_x^{1,0} = Q_y^{1,0} = 0$, that the z-components are symmetric with respect to u, while the y- and x-components of all quantities are antisymmetric, and, moreover, that $j_x^{p,k}$, $j_z^{p,k}$, $H_y^{p,k}$, $Q_x^{p,k}$, $Q_z^{p,k}$ contain

only even harmonics, while the remaining quantities have only odd ones. Therefore, in what follows the second upper index will denote not the harmonic number, but the ordinal number of the nonzero harmonics.

At each successive stage the following calculations are carried out:

1. For a given current in the plasma, the magnetic field is determined from the linear system of equations

$$dH_y^{1,0}/du = j_z^{1,0}, \qquad (2k-1)\beta(-1)^p H_x^{p,k} - dH_z^{3-p,k}/du = j_y^{3-p,k}, \tag{4}$$

$$dH_x^{p,k}/du - (-1)^p(2k-1)\beta H_z^{3-p,k} = 0 \qquad (p = 1, 2)$$

with the boundary conditions

$$H_x^{1,k} = H_z^{2,k}, \qquad H_x^{2,k} + H_z^{1,k} = -\Delta e_{1k}^{-\beta\delta}, \tag{5}$$

where $\Delta = e^{-\gamma\beta}$. The boundary conditions (5) are obtained from the solution in the external region and make it possible to consider the posed problem only inside the layer. The solution of system (4) has the form

$$H_y^{1,0} = \int_0^u j_z^{1,0}(t)\,dt,$$

$$H_x^{p,k} = \Delta\ \mathrm{sh}\,\beta u\,\delta_{k1}\delta_{p2} - (-1)^p R_1\{j_y^{3-p,k}\},$$

$$H_z^{p,k} = \Delta\ \mathrm{ch}\,\beta u\,\delta_{k1}\delta_{p1} + R_2\{j_y^{p,k}\}, \qquad (p=1,2), \tag{6}$$

where

$$R_{1,2}\{f\} = -\psi(u)\int_0^u \varphi_1(t)f(t)\,dt \mp \varphi_{1,2}(u)\int_u^1 \psi(t)f(t)\,dt, \quad \psi(x) = e^{-(2k-1)\beta x},$$

$$\varphi_1 = \mathrm{sh}(2k-1)\beta x, \qquad \varphi_2 = \mathrm{ch}(2k-1)\beta x.$$

2. Using the obtained $H_x^{p,k}$, $H_z^{p,k}$ and the prescribed $j^{p,k}$, the $Q^{p,k}$ are determined:

$$Q_z^{1,0} = \frac{\Omega}{2}\sum_n \sum_{p=1,2} j_y^{p,n} H_x^{p,n},$$

$$\begin{aligned} Q_x^{p,k} = {} & \frac{(-1)^p}{2k\beta}(1-\Omega j_z^{1,0}) j_x^{3-p,k} + \Omega H_y^{1,0} j_z^{p,k} \\ & - \frac{\Omega}{2}\sum_n \Big\{ j_y^{p,n}\big[H_z^{1,\nu} - (-1)^p H_z^{1,\lambda}\big] + j_y^{3-p,n}\big[(-1)^p \,\mathrm{sign}(k-n+\tfrac{1}{2}) H_z^{2,\nu} + H_z^{2,\lambda}\big] \\ & \qquad - \frac{1}{2\beta x}\big[j_z^{p,n} j_x^{2,x} + (-1)^p\,\mathrm{sign}(n-k) j_z^{2,n} j_x^{p,x}\big]\Big\}, \end{aligned} \tag{7}$$

etc., where $\nu = |n-k-\frac{1}{2}| + \frac{1}{2}$, $\lambda = n+k$, $x = |n-k|$. In these formulas essentially all nonlinear effects are contained.

3. From the $Q^{p,k}$ obtained in item 2, the components of the currents in the plasma are calculated for the next approximation. Taking (3) into account, equations (2) take the form:

$$j_y^{p,k} = \alpha Q_y^{p,k}, \qquad \frac{d}{du}(j_z^{1,0} - \alpha Q_z^{1,0}) = 0,$$

$$2k\beta j_x^{p,k} - (-1)^p d j_z^{3-p,k}/du = \alpha\big[2k\beta Q_x^{p,k} - (-1)^p dQ_z^{3-p,k}/du\big], \tag{8}$$

$$dj_x^{p,k}/du - (-1)^p 2k\beta j_z^{3-p,k} = 0 \qquad (p = 1, 2)$$

with the boundary condition $j_x^{p,k} = 0$ at $u = 1$. System (8) is readily solved:

$$j_x^{p,k} = \frac{2\alpha k\beta}{\Phi_1(1)} \left\{ \Phi_1(1-u) \int_0^u \left[Q_x^{p,k}\Phi_1(t) + (-1)^p Q_z^{3-p,k}\Phi_2(t)\right] dt \right.$$
$$\left. + \Phi_1(u) \int_u^1 \left[Q_x^{p,k}\Phi_1(1-t) - (-1)^p Q_z^{3-p,k}\Phi_2(1-t)\right] dt \right\}, \qquad (9)$$

$$j_z^{p,k} = \alpha(-1)^p Q_z^{p,k} + (-1)^p \frac{2\alpha k\beta}{\Phi_1(1)} \left\{ \Phi_2(1-u) \int_0^u \left[Q_x^{3-p,k}\Phi_1(t) - \right.\right.$$
$$-(-1)^p Q_z^{p,k}\Phi_2(t)\Big] dt - \Phi_2(u) \int_u^1 \Big[Q_x^{3-p,k}\Phi_1(1-t) +$$
$$\left. + (-1)^p Q_z^{p,k}\Phi_2(1-t)\Big] dt \right\} \qquad (p = 1, 2),$$

where $\Phi_1(x) = \operatorname{sh} 2k\beta x$, $\Phi_2(x) = \operatorname{ch} 2k\beta x$.

Thus the solution of the system of nonlinear partial differential equations has been reduced to the simplest operations of integration and the computation of sums, repeated many times. If one assumes that at the initial stage the field in the system has the form

$$H_x^{1,1} = \Delta \operatorname{sh} \beta u, \qquad H_z^{2,1} = \Delta \operatorname{ch} \beta u \qquad (10)$$

(the field of an empty system), then all the integrals in (6) and (9) are evaluated in quadratures. Therefore, in principle it is not difficult to obtain exact analytic expressions for all quantities at any prescribed stage in advance; however, the practical value of the final formulas would hardly justify the effort expended, since in order to obtain intelligible results one must in any case use a computer. It is expedient from the very beginning to carry out all calculations (including the integration) on a computer according to the proposed scheme, using (6), (7), and (9). A convenient check on the validity of the method and on convergence is the calculation by this method of the ordinary skin effect for a plasma layer in the field of a traveling wave of an external current, i.e., for $\Omega = 0$, and comparison of the obtained solution with the analytic one.

The results of the computer calculation fully confirmed the correctness of the approach used to solve the problem. With the aid of the obtained quantities

Table 1*

α	13.7	13.7	13.7	27.3	27.3	68.5	68.5	68.5	68.5	137	137
Ω	0	0.5	1.0	0	0.5	0	0.1	0.2	0.5	0	0.5
I	0	0.23	0.39	0	0.39	0	0.14	0.28	0.70	0	0.85
P	0.034	0.033	0.029	0.028	0.029	0.020	0.021	0.021	0.021	0.015	0.015
$\mathcal{H}_z$	0.57	0.52	0.45	0.61	0.54	0.64	0.63	0.62	0.51	0.66	0.49
$\mathcal{H}_x$	0.17	0.20	0.26	0.13	0.18	0.09	0.095	0.11	0.19	0.07	0.22
d/x	0.30	0.40	0.58	0.21	0.37	0.14	0.16	0.17	0.40	0.10	0.40

* $I = I_z/2i_0$, $P = \gamma^2 c^2 \mathcal{P}/16\pi^2 i_0^2 \omega$, $\mathcal{H}_{x,z}$—maximum values of $H_{x,z}$ at the boundary, d—the depth at which the magnetic field decreases by a factor of e.

$j^{p,k}$, $\mathbf{H}^{p,k}$, one can compute the distributions of magnetic fields and currents in the plasma that are of physical interest, the total entrainment current I_z, and the active line losses $\mathcal{P}$. Table 1 gives the results of numerical calculations for several values of the parameters α and Ω at fixed values $\beta = 1.37$ and $\Delta = 0.18$, corresponding to $a = 4$ cm, $b = 5$ cm, $\omega = 1.5 \cdot 10^7$ s^{-1}, $\gamma = 0.34$ cm^{-1}. It is seen that the depth of penetration of the field into the plasma increases with increasing Ω, owing to the decrease in the effective conductivity of the plasma across the magnetic field. An entrainment current arises, and its density at no point of the cross section reaches the maximum value corresponding to complete entrainment of the electrons in the plasma. The magnitude of the active losses changes only slightly when Ω changes, although a substantial redistribution occurs of the contributions to them from the different current components: at small Ω the main contribution to the losses is made by the component j_y, whereas at large Ω it is made by the longitudinal component of the current j_z.

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References

1. V. S. Imshennik, *Some nonlinear problems of the dynamics of a dense high-temperature plasma*, Dissertation, Moscow, 1967.

Note: Figure translations are in progress. See original paper for figures.