ON THE NUMERICAL SOLUTION OF ILL-POSED PROBLEMS REPRESENTED BY INTEGRAL EQUATIONS OF CONVOLUTION TYPE

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Abstract

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MATHEMATICS

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ON THE NUMERICAL SOLUTION OF ILL-POSED PROBLEMS REPRESENTED BY INTEGRAL EQUATIONS OF CONVOLUTION TYPE

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Many problems of exploration geophysics ([1,2] and others) lead to linear integral equations of convolution type

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \varphi(\xi) K(x-\xi)\, d\xi, \qquad -\infty \leq x \leq +\infty, \tag{1}$$

whose kernels satisfy the conditions:

$$\begin{aligned} &1) \quad |k(t)| \leq 1, \qquad -\infty \leq t \leq +\infty; \\ &2) \quad k(t) \neq 0 \text{ for almost all } t, \; -\infty < t < +\infty. \end{aligned} \tag{2}$$

Here

$$k(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} K(x) e^{-itx}\, dx \tag{3}$$

is the spectrum (Fourier transform) of the kernel $K(x)$.

We write equation (1) in operator form:

$$f = A\varphi; \qquad \|A\| = \sup_t |k(t)| = 1, \tag{4}$$

where A is a linear integral operator acting in the space L^2, from the set P, singled out by condition 2); f and φ are functions from the space L^2. Obviously, there exists an inverse operator A^{-1} to A, but

$$\|A^{-1}\| = \sup_t |1/k(t)| = +\infty, \tag{5}$$

so that the solutions of equation (1) possess instability: small (in the norm in L^2) variations δf may correspond to large variations $\delta\varphi$.

A. N. Tikhonov (^{3,4}) proposed a method for the approximate solution of problems of the type (4)–(5)—the *regularization method*. It consists in introducing, instead of equation (4), a linear operator T_α in the space L^2, depending on a parameter α, $0 < \alpha < \alpha_0$, and satisfying the conditions:

$$\begin{aligned} &1)\quad \|T_\alpha\| < +\infty, \qquad 0 < \alpha < \alpha_0; \\ &2)\quad \|\varphi - T_\alpha A\varphi\| \xrightarrow[\alpha\to 0]{} 0 \quad (\text{for any } \varphi \in L^2). \end{aligned} \tag{6}$$

(7)

The function $T_\alpha f$, for some $\alpha = \alpha^*$, is taken as an approximate solution of equation (4). V. K. Ivanov (^{5}) introduced the notion of uniform regularization. Namely, the operator T_α uniformly regularizes equation (4) on some set $m \subseteq L^2$, if, as $\alpha \to 0$, $\|\varphi - T_\alpha A\varphi\| \to 0$ uniformly for all $\varphi \in m$.

If an equation $f_\delta = A\varphi$ is given, where $f_\delta = f + \delta f$, $\|\delta f\| \le \delta$, and moreover $f \in M$, $\delta f \in M$, then $f_\delta \in M$, and a solution of the equation does not exist for any $\delta > 0$. However, an approximate solution can always be constructed, pro-

than

$$\|\varphi - T_\alpha f_\delta\| \le \|\varphi - T_\alpha A\varphi\| + \|T_\alpha\|\delta, \tag{8}$$

and by an appropriate choice of α one can achieve sufficiently good results.

Definition 1. The value of the parameter α determined from the condition

$$\sup_{\varphi\in m} \|\varphi - T_\alpha A\varphi\| + \|T_\alpha\|\delta = \min \tag{9}$$

is called **optimal on the set m of uniform regularization for a given** $\delta > 0$.

Accordingly, $T_\alpha f_\delta$, where α is chosen according to (9), will be an optimal (on m for the given δ) solution.

Definition 2. A linear operator L, $\|L\| < +\infty$, acting in the space L^2, will be called **computationally finite-dimensional** if, in order to find the function Lf, $f(x) \in L^2$, at one value of the argument x, a finite number of values $f(x)$ is used, on which a finite number of arithmetic operations is performed. Operators L of the opposite kind will be called **computationally infinite-dimensional**.

If T_α is a regularizing operator, but computationally infinite-dimensional, then constructing the function $T_\alpha f$ even for a single x is practically impossible. Therefore one proceeds as follows: the operator T_α is approximately approximated by some computationally finite-dimensional operator $\widetilde{T}_\alpha$, and the approximate solution of the problem is taken to be not $T_\alpha f$, but $\widetilde{T}_\alpha f$. However, if the method for constructing the operator $\widetilde{T}_\alpha$ from the operator T_α is not regular, in a certain way connected with the value of α used, then convergence may be lost: although $T_\alpha A\varphi \to \varphi$ as $\alpha \to 0$, it may be that $\widetilde{T}_\alpha A\varphi \to \widetilde{\varphi}$, $\|\varphi - \widetilde{\varphi}\| > 0$. Hence it follows that the procedure of passing from the computationally infinite-dimensional operator T_α to the computationally finite-dimensional $\widetilde{T}_\alpha$ must also be regularized. Such an operation is most simply carried out by introducing a second parameter $\beta = n$ (n natural) and putting $\widetilde{T}_\alpha = T_{\alpha,n}$, where, as $n \to \infty$, $T_{\alpha,n} f \to T_\alpha f$ for any $f \in L^2$.

Definition 3. A linear operator $T_{\alpha,n}$, $0 \le \alpha \le \alpha_0$, $n_0 \le n \le +\infty$, is called **regularizing equation (4) on the space** L^2 if: 1) for all n and α, except the case when simultaneously $n = +\infty$ and $\alpha = 0$,

$$\|T_{\alpha,n}\| < +\infty; \tag{10}$$

2) if at least for one sequence of values α and n

$$\|\varphi - T_{\alpha,n} A\varphi\| \xrightarrow[\alpha\to 0,\ n\to\infty]{} 0 \quad \text{for any } \varphi \in L^2. \tag{11}$$

Definition 4. The operator $T_{\alpha,n}$ is called **uniformly regularizing equation (4) on the set** $m \in L^2$ if, for all $\varphi \in m$ and at least for one sequence (α, n), $\|\varphi - T_{\alpha,n} A\varphi\| \to 0$ uniformly.

Definition 5. The values of the parameters α and n, determined from the condition

$$\sup_{\varphi\in m} \|\varphi - T_{\alpha,n} A\varphi\| + \|T_{\alpha,n}\|\delta = \min, \tag{12}$$

are called **optimal on the set** m **of uniform regularization for a given** $\delta > 0$.

Theorem 1. *In order that the two-parameter operator* $T_{\alpha,n}$ *be regularizing, it is sufficient that* $T_{\alpha,\infty} = T_\alpha$ *be a one-parameter regularizing operator and*

$$\|T_\alpha - T_{\alpha,n}\| \xrightarrow[n\to\infty]{} 0 \tag{13}$$

uniformly on any interval $0 < \alpha_0^* \le \alpha \le \alpha_0$.

Proof follows from the inequality

$$\|\varphi - T_{\alpha,n}A\varphi\| \leq \|\varphi - T_\alpha A\varphi\| + \|T_\alpha - T_{\alpha,n}\| \, \|A\varphi\|. \tag{14}$$

Let us proceed to the construction of methods for the approximate solution of integral equations of convolution type (1)–(4) by means of computationally finitely supported two-parameter regularizing operators. We shall seek the latter in the class of operators

$$S_{\alpha,n} = \sum_{-n}^{+n} c_k E_\alpha^k, \tag{15}$$

where E_α is the shift operator, $E_\alpha^k\{f(x)\} = f(x + k\alpha)$, and the c_k are numerical coefficients depending on α and n. The approximate solution $\varphi_{\alpha,n}(x)$ of equation (1) on the basis of the operator $S_{\alpha,n}$ is determined by the expression

$$\varphi_{\alpha,n}(x) = \sum_{-n}^{+n} c_k f(x + k\alpha). \tag{16}$$

Clearly, under the conditions for the existence of a solution $\varphi(x) \in L^2$ of equation (1),

$$\|\varphi - \varphi_{\alpha,n}\|_{L_2} = \left(\int_{-\infty}^{+\infty} |F(t)|^2 \left| \frac{1}{k(t)} - \sum_{-n}^{+n} c_k e^{-ik\alpha t} \right|^2 dt \right)^{1/2}, \tag{17}$$

where $F(t)$ is the Fourier transform of the function $f(x)$.

A natural method is the choice of coefficients from the conditions of expanding the function $1/k(t)$ in a Fourier series on the interval $|t| \leq \pi/\alpha$:

$$c_k = c_k(\alpha) = \frac{\alpha}{2\pi} \int_{-\pi/\alpha}^{+\pi/\alpha} \frac{e^{ik\alpha t}}{k(t)} \, dt, \qquad k = 0, \pm 1, \pm 2, \ldots \tag{18}$$

Consider the subclass of integral equations (1)–(4) satisfying the additional conditions, fulfilled for all $\alpha > 0$:

$$\mu(\alpha) = \sup_{|t| \leq \pi/\alpha} \left| \frac{1}{k(t)} \right| < +\infty; \tag{19}$$

$$|1 - k(t)/k_{\text{per}}^{(\alpha)}(t)| \leq C, \qquad |t| > \pi/\alpha, \tag{20}$$

where C is an absolute constant, and $k_{\text{per}}^{(\alpha)}(t)$ is the $2\pi/\alpha$-periodic repetition of the values of the function $k(t)$ on the interval $|t| \leq \pi/\alpha$.

Theorem 2. *In order that, under conditions (19)–(20), the operator* $S_{\alpha,n}$ *(15) with coefficients (18) be regularizing, it is sufficient that the expansion*

$$\frac{1}{k_{\text{per}}^{(\alpha)}(t)} = \sum_{-\infty}^{+\infty} c_k(\alpha) e^{-ik\alpha t}, \qquad -\infty \le t \le +\infty, \tag{21}$$

converge uniformly for all $\alpha > 0$*.*

Proof. The operator $S_{\alpha,n}$ is bounded for arbitrary $\alpha > 0$, $n < +\infty$:

$$\|S_{\alpha,n}\| = \sup_t \left| \sum_{-n}^{+n} c_k e^{-ik\alpha t} \right| < +\infty. \tag{22}$$

The operator $S_\alpha = S_{\alpha,\infty}$ is regularizing. Indeed,

$$\|S_\alpha\| = \sup_{\|f\| \le 1} \left(\int_{-\infty}^{+\infty} |F(t)|^2 \left| \frac{1}{k_{\text{per}}^{(\alpha)}(t)} \right|^2 dt \right)^{1/2} = \sup_{|t| \le \pi/2} \left| \frac{1}{k(t)} \right| = \mu(\alpha) < +\infty \tag{23}$$

and ($\Phi(t)$ is the Fourier transform of the function $\varphi(x)$):

$$\|\varphi - S_\alpha A \varphi\| = \left(\int_{-\infty}^{+\infty} |F(t)|^2 \left| \frac{1}{k(t)} - \frac{1}{k_{\text{per}}^{(\alpha)}(t)} \right|^2 dt \right)^{1/2} \le$$

$$\le C \left(\int_{-\infty}^{-\pi/\alpha} |\Phi(t)|^2 \, dt + \int_{\pi/\alpha}^{+\infty} |\Phi(t)|^2 \, dt \right)^{1/2}, \tag{24}$$

and since

$$\int_{-\infty}^{+\infty} |\Phi(t)|^2 \, dt < +\infty,$$

as $\alpha \to 0$ the right-hand side in (24) tends to zero. Since, in addition,

$$\begin{aligned} \|S_\alpha - S_{\alpha,n}\| &= \sup_{\|f\| \le 1} \left(\int_{-\infty}^{+\infty} |F(t)|^2 \left| \frac{1}{k_{\text{per}}^{(\alpha)}(t)} - \sum_{-n}^{+n} c_k e^{-ikat} \right|^2 dt \right)^{1/2} \\ &= \sup_{|t| \le \pi/\alpha} \left| \frac{1}{k_{\text{per}}^{(\alpha)}(t)} - \sum_{-n}^{+n} c_k e^{-ikat} \right|, \end{aligned} \tag{25}$$

and, by virtue of the uniform convergence (21), tends to zero as $n \to \infty$, the validity of Theorem 2 follows from Theorem 1.

Let $\varphi_\sigma(x) \in L^2$ be a collection of functions whose spectra $\Phi_\sigma(t)$ satisfy the boundedness condition:

$$\Phi_\sigma(t) \equiv 0 \quad \text{for } |t| > \sigma. \tag{26}$$

The set m of functions φ on which the two-parameter regularizing operator $S_{\alpha,n}$ admits uniform regularization can be specified as the collection of functions with bounded spectrum (26), with $\|\varphi_\sigma\| \leq N$.

In this case, for $\varphi = \varphi_\sigma \in m$, we have the estimate

$$\|\varphi_\sigma - S_{\alpha,n} A \varphi_\sigma\| \leq N \sup_{|t| \leq \sigma} \left| \frac{1}{k_{\text{per}}^{(\alpha)}(t)} - \sum_{-n}^{+n} c_k e^{-ikat} \right| + \delta \sup_{|t| \leq \sigma} \left| \sum_{-n}^{+n} c_k e^{-ikat} \right| . \tag{27}$$

The operator $S_{\alpha,n}$ optimal on the set $\varphi_\sigma \in m$, for a given δ, is determined by the condition

$$\frac{\delta}{N} \sup_{|t| \leq \sigma} \left| \sum_{-n}^{+n} c_k e^{-ikat} \right| + \sup_{|t| \leq \sigma} \left| \frac{1}{k_{\text{per}}^{(\alpha)}(t)} - \sum_{-n}^{+n} c_k e^{-ikat} \right| = \min . \tag{28}$$

Up to now we have considered regularization on the whole space L^2. However, regularization may also be considered on certain sets E of functions from L^2 admitting a stronger metrization. In these cases, the operator regularizing on the set E is defined analogously, with the norm from L^2 in (7) and (11) replaced by the stronger norm of this set.

Theorem 3. Let $L^2_{(k+1)}$ be the set of functions differentiable $k+1$ times, with derivatives belonging to L^2. Under the condition $\varphi(x) \in L^2_{(k+1)}$ and the fulfillment of the restrictions (19)–(21), the operator $S_{\alpha,n}$ is regularizing in the sense of uniform convergence

$$\max_x \left| d^r \varphi(x)/dx^r - d^r \varphi_{\alpha,n}(x)/dx^r \right| \longrightarrow 0, \quad r \leq k, \tag{29}$$

as $\alpha \to 0,\ n \to \infty$, and convergence

$$\left\| d^{k+1} \varphi(x)/dx^{k+1} - d^{k+1} \varphi_{\alpha,n}(x)/dx^{k+1} \right\| \to 0$$

in the space L^2.

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