ON HILL STABILITY IN THE UNRESTRICTED THREE-BODY PROBLEM

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Abstract

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ON HILL STABILITY IN THE UNRESTRICTED THREE-BODY PROBLEM

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§ 1. We shall consider the motion of three bodies P_1, P_2, P_3 with masses m_1, m_2, m_3, attracting one another according to Newton' s law, in a barycentric system of coordinates ([1]). The energy integral has the form $T = U + h$, where T is the kinetic energy of the system,

$$U = f(m_1 m_2 r_{12}^{-1} + m_1 m_3 r_{13}^{-1} + m_2 m_3 r_{23}^{-1})$$

is the force function (f is the constant of attraction, r_{ij} is the distance between P_i and P_j), and h is the energy constant.

We shall consider only the case $h < 0$, for which it is convenient to introduce the constant $h' = -h > 0$, so that the energy integral takes the form $T = U - h'$. Since $T \geqslant 0$,

$$\min_{i \neq j} r_{ij} \leqslant f(m_1 m_2 + m_1 m_3 + m_2 m_3)/h'.$$

But from this there still does not follow Hill stability of even a single pair of bodies, i.e., boundedness from above of the distance between two specified bodies on an unbounded time interval. We shall rely on the known inequality

$$I(U - h') \geqslant \frac{1}{2} C^2, \tag{1}$$

where

$$I = M^{-1}(m_1 m_2 r_{12}^2 + m_1 m_3 r_{13}^2 + m_2 m_3 r_{23}^2)$$

is the quadratic moment of inertia of the system relative to the barycenter ($M = m_1 + m_2 + m_3$), and C is the modulus of the constant vector of angular momentum (we shall consider the principal case $C \neq 0$, i.e. $C > 0$). Inequality

(1) is a consequence of the known Sundman inequality ([2]), but it can also be justified independently of the latter. From (1) it follows that

$$IU^2 \geqslant 2h'C^2. \tag{2}$$

Indeed, consider the quadratic trinomial

$$\varphi(x) = h'Ix^2 - IUx + C^2/2.$$

Since, by virtue of (1), $\varphi(1) \leqslant 0$, while $h'I > 0$ ($I > 0$, since by Sundman' s theorem ([2]) always $I \neq 0$ in view of $C \neq 0$), the discriminant of $\varphi(x)$ is nonnegative, which leads to (2). It can be shown that the equality sign can be attained under certain conditions both in (1) and in (2). The left-hand side of (2) is a homogeneous function of degree zero with respect to the mutual distances; therefore it depends only on the shape of the triangle $P_1P_2P_3$, and not on its dimensions. One may, for example, regard it as depending only on $\rho_1 = r_{13}/r_{12}$ and $\rho_2 = r_{23}/r_{12}$, i.e., on the relative distances of P_3 from P_1 and P_2. If we introduce the relative masses $\mu_j = m_j/M$, $j = 1, 2, 3$ ($0 < \mu_j < 1$, $\mu_1 + \mu_2 + \mu_3 = 1$), and divide (2) by f^2M^5, the latter inequality can be reduced to the form

$$S \geqslant s,$$

where

$$s = 2h'C^2/f^2M^5$$

is a constant dimensionless quantity, which we shall call the stability index, and

$$S = iu^2,$$

where

$$i = \mu_1\mu_2 + \mu_1\mu_3\rho_1^2 + \mu_2\mu_3\rho_2^2, \qquad u = \mu_1\mu_2 + \mu_1\mu_3\rho_1^{-1} + \mu_2\mu_3\rho_2^{-1},$$

so that $S = S(\rho_1, \rho_2)$. Introduce in the current plane $P_1P_2P_3$ a right rectangular coordinate system $O_{12}\xi\eta$ in the following way. Take as unit of scale the current distance $P_1P_2 = r_{12}$, and as origin the center of mass O_{12} of the bodies P_1 and P_2; draw the axis $O_{12}\xi$ through P_1 and P_2, from P_1 to P_2. Thus P_1, P_2, and P_3 will have the following coordinates:

$$P_1(\xi_1, 0), \quad P_2(\xi_2, 0), \quad P_3(\xi, \eta),$$

where

$$\xi_1 = -\mu_2/(\mu_1+\mu_2), \qquad \xi_2 = \mu_1/(\mu_1+\mu_2)$$

$$(\xi_2 - \xi_1 = 1);$$

moreover,

$$\rho_1 = [(\xi-\xi_1)^2+\eta^2]^{1/2}, \qquad \rho_2 = [(\xi-\xi_2)^2+\eta^2]^{1/2},$$

so that S ultimately turns out to be a complicated function of ξ and η.

§ 2. The inequality $S \geq s$ determines that region of the plane $O_{12}\xi\eta$ in which P_3 can move. Under our assumptions, the stability index s, determined by the initial conditions, can have any value in the range $0 < s < +\infty$. If s is sufficiently large, then the set of points of the curve $S(\xi,\eta) = s$ is nonempty, and the curve serves as the boundary of the region in which the motion of P_3 is possible.

The problem is thus reduced to the study of the family of curves $S(\xi,\eta) = S_0 > 0$. If S_0 is very large, then the curve evidently consists of three closed branches —ovals, close in form to circles: two small ones enclosing the point P_1 or P_2, and one large one enclosing both P_1 and P_2. As S_0 decreases, the inner ovals increase, while the outer one decreases. The meeting of different branches of the curve $S(\xi,\eta) = S_0$, or their disappearance, can occur only at points where $\partial S/\partial\xi = \partial S/\partial\eta = 0$ (singular points of the family of curves $S = \text{const}$). Certain critical values of the parameter S_0 correspond to these points. Thus we arrive at the system of equations

$$\frac{\partial S}{\partial\lambda} = \frac{\partial S}{\partial\rho_1}\frac{\partial\rho_1}{\partial\lambda} + \frac{\partial S}{\partial\rho_2}\frac{\partial\rho_2}{\partial\lambda} = 0, \qquad \lambda = \xi,\eta, \tag{3}$$

where

$$\frac{\partial\rho_k}{\partial\xi} = \frac{\xi-\xi_k}{\rho_k}, \qquad \frac{\partial\rho_k}{\partial\eta} = \frac{\eta}{\rho_k}, \qquad k = 1,2, \tag{4}$$

$$\frac{\partial S}{\partial\rho_k} = 2\mu_1\mu_2\mu_3\nu\left[\mu_k\left(\rho_k - \frac{1}{\rho_k^2}\right) + \mu_3\left(\frac{\rho_k}{\rho_{3-k}} - \frac{\rho_{3-k}^2}{\rho_k^2}\right)\right], \qquad k = 1,2. \tag{5}$$

Let us first find the singular points for which $\eta \neq 0$, i.e., those not lying on the axis $O_{12}\xi$. The determinant of system (3) with respect to the unknowns $\partial S/\partial\rho_1$ and $\partial S/\partial\rho_2$ is reduced to $\eta/\rho_1\rho_2$, i.e., is not equal to 0. Therefore (3) becomes the system $\partial S/\partial\rho_1 = \partial S/\partial\rho_2 = 0$. Equating the square brackets in (5), for $k = 1, 2$, to zero and solving the resulting system with respect to ρ_1 and ρ_2, we find $\rho_1 = \rho_2 = 1$. This gives two singular points, located at the vertices of two equilateral triangles with base P_1P_2. Denoting them by L_4 and L_5, we obtain $L_{4,5}((\xi_1 + \xi_2)/2, \pm\sqrt{3}/2)$.

On the axis $O_{12}\xi$ it is necessary to consider three intervals: 1) $-\infty < \xi < \xi_1$, 2) $\xi_1 < \xi < \xi_2$, and 3) $\xi_2 < \xi < +\infty$. Since on the entire axis $\partial S/\partial\eta|_{\eta=0} \equiv 0$, it is only necessary to solve the equation $\partial S/\partial\xi|_{\eta=0} = 0$. On the first interval $\rho_1 = \xi_1 - \xi, \rho_2 = \xi_2 - \xi$, on the second $\rho_1 = \xi - \xi_1, \rho_2 = \xi_2 - \xi$, and on the third $\rho_1 = \xi - \xi_1, \rho_2 = \xi - \xi_2$. Therefore we arrive at the equations: on the first interval $\partial S/\partial\xi \equiv -(\partial S/\partial\rho_1 + \partial S/\partial\rho_2) = 0$, on the second $\partial S/\partial\xi \equiv \partial S/\partial\rho_1 - \partial S/\partial\rho_2 = 0$, and on the third $\partial S/\partial\xi \equiv \partial S/\partial\rho_1 + \partial S/\partial\rho_2 = 0$. Instead of ξ on these intervals we introduce variables: on the first $\alpha_1 = 1/\rho_1$, on the second $\alpha_2 = \rho_2/\rho_1$, on the third $\alpha_3 = \rho_2$. Obviously $0 < \alpha_j < +\infty$, $j = 1, 2, 3$. As a result, on the first interval we arrive at the equation

$$(\mu_1+\mu_3)\alpha_1^5+(2\mu_1+3\mu_3)\alpha_1^4+(\mu_1+3\mu_3)\alpha_1^3-(\mu_1+3\mu_2)\alpha_1^2-(2\mu_1+3\mu_2)\alpha_1-(\mu_1+\mu_2) = 0.$$

The equation for the second interval is obtained from this by replacing α_1 by α_2, μ_1 by μ_3, and μ_3 by μ_1; for the third by replacing α_1 by α_3, μ_1 by μ_2, μ_2 by μ_3, and μ_3 by μ_1. Each of these equations has one change of sign and, consequently, a unique positive simple root (by Descartes' theorem). Thus in the j-th interval, $j = 1, 2, 3$, there is a unique singular point, which we shall denote by L_j. All five singular points, according to (3), coincide with the libration points of the body P_3 relative to the bodies P_1 and P_2, which justifies their notation.

§ 3. To determine the character of the singular points it is necessary to determine in them the sign of the expression $\Delta = (\partial^2 S/\partial\xi^2)(\partial^2 S/\partial\eta^2) - (\partial^2 S/\partial\xi\partial\eta)^2$. Direct calculation shows that $\Delta(L_4) = \Delta(L_5) = 27\mu_1^2\mu_2^2\mu_3^2(\mu_1\mu_2+\mu_1\mu_3+\mu_2\mu_3)^3 > 0$, and moreover $\partial^2 S/\partial\xi^2 > 0$. Consequently, $S(\xi, \eta)$ has at the points L_4 and L_5 a minimum, which is equal to $(\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3)^3$. The curve

$S(\xi, \eta) = S(L_4) = S(L_5)$ degenerates into two isolated points L_4 and L_5.

Let us turn to the special points L_1, L_2, L_3. First of all, on the entire axis $O_{12}\xi$, $\partial^2 S/\partial\xi\partial\eta\big|_{\eta=0} \equiv 0$. An analysis, the details of which we cannot give here for lack of space, shows that $\partial^2 S/\partial\xi^2\big|_{L_j} > 0$, while $\partial^2 S/\partial\eta^2\big|_{L_j} < 0$, $j = 1, 2, 3$. Thus, $\Delta(L_j) < 0$, $j = 1, 2, 3$. Consequently, L_j is a saddle point for the function $S(\xi, \eta)$ and a special point of self-intersection (a node) with two tangents for the curve $S(\xi, \eta) = S(L_j)$. Hence, and from the fact that S tends to $+\infty$ at the points P_1, P_2 and at the infinitely distant point, it is not difficult to derive that S has at the points L_4 and L_5 not only a local but also a global minimum.

Denote $S(\xi, 0)$ by $S_1(\xi)$ for $-\infty < \xi < \xi_1$, by $S_2(\xi)$ for $\xi_1 < \xi < \xi_2$, and by $S_3(\xi)$ for $\xi_2 < \xi < +\infty$. Since $dS_j/d\xi \equiv \partial S/\partial\xi\big|_{\eta=0} = 0$ only at the point L_j

of the entire j-th interval, $j = 1, 2, 3$, and $d^2S_j/d\xi^2 \equiv \partial^2 S/\partial\xi^2\big|_{\eta=0} > 0$ at the point L_j, it follows that on the j-th interval $S_j(\xi)$ decreases to the left of L_j and increases to the right of L_j, while at the point L_j itself $S_j(\xi)$ attains a minimum (at the ends of the j-th interval, S_j, as noted, tends to $+\infty$).

§ 4. An important question now is the comparison of the quantities $S(L_j)$, $j = 1, 2, 3$, with one another (the value $S(L_4) = S(L_5)$, as already noted, is certainly smaller than these quantities). For this purpose we shall establish one important proposition, which we shall need twice. Let, for definiteness, $\mu_1 \geqslant \mu_2$. Introduce polar coordinates by the formulas $\xi = \rho\cos\theta$, $\eta = \rho\sin\theta$, and consider $S(\rho, \theta)$, assuming that $\rho = \text{const} > \xi_2 \geqslant |\xi_1|$. It is easy to see that i does not depend on θ, so that as θ varies S increases or decreases simultaneously with u, i.e., simultaneously with the quantity $\nu = \mu_1\rho_1^{-1} + \mu_2\rho_2^{-1}$. From symmetry considerations it is sufficient to restrict ourselves to the segment $0 \leqslant \theta \leqslant \pi$. At only one point of the interval $0 < \theta < \pi$ is $d\nu/d\theta = 0$: when $\rho_1 = \rho_2$; moreover, ν has a minimum there. Consequently, ν assumes its greatest value at one of the ends of the segment. It turns out that for $\mu_1 \geqslant \mu_2$, $\nu(\rho, 0) \geqslant \nu(\rho, \pi)$. Thus, for $\mu_1 \geqslant \mu_2$ and $\rho > \xi_2 \geqslant |\xi_1|$, $S(\rho, \theta) \leqslant S(\rho, 0) = S_3(\rho)$. Here $S(\rho, \pi) = S(\rho, 0)$ only when $\mu_1 = \mu_2$.

Now we are able to compare the quantities $S(L_j)$, $j = 1, 2, 3$. Let us give the final results. Suppose the bodies are numbered so that $\mu_1 \geqslant \mu_2 \geqslant \mu_3$. Then: 1) if $\mu_1 > \mu_2 > \mu_3$, then $S(L_2) > S(L_3) > S(L_1) > S(L_4) = S(L_5)$; 2) if $\mu_1 = \mu_2 > \mu_3$, then $S(L_2) > S(L_3) = S(L_1) > S(L_4) = S(L_5)$; 3) if $\mu_1 > \mu_2 = \mu_3$, then $S(L_2) = S(L_3) > S(L_1) > S(L_4) = S(L_5)$; 4) if $\mu_1 = \mu_2 = \mu_3 = 1/3$, then $S(L_2) = S(L_3) = S(L_1) > S(L_4) = S(L_5)$.

Let us examine in detail the most general first case. If $s > S(L_2)$, then the curve $S(\xi, \eta) = s$ consists of three separate ovals, and P_3 may move, in view of $S \geqslant s$, either inside the oval enclosing P_1, or inside the oval enclosing P_2, or outside the outer oval—depending on where it was at the initial instant. In the first case the bodies P_1 and P_3 will be Hill-stable, in the second P_2 and P_3, and in the third P_1 and P_2. Indeed, if, for example, in the first case $P_1P_3 = r_{13}$ could become arbitrarily large, then simultaneously with it the two other distances would also become arbitrarily large, which contradicts the boundedness of the smallest of the mutual distances following from the energy integral. If $S(L_2) > s > S(L_3)$, then $S(\xi, \eta) = s$ consists of only two branches: inside the outer oval, instead of two small ovals, there is one closed line, resembling in shape an hourglass in longitudinal section and enclosing P_1 and P_2. The body P_3 may move either inside this line or outside the outer oval. In the latter case P_1 and P_2 are still Hill-stable; in the former case, generally speaking,

one cannot guarantee Hill stability of any pair of bodies. If $S(L_3) > s$, then the curve $S(\xi, \eta) = s$ consists of one horseshoe-shaped branch for $s > S(L_1)$, or of two lobes enclosing L_4 and L_5, for $S(L_1) > s > S(L_4) = S(L_5)$. In this case, Hill stability of any pair of bodies can no longer be guaranteed.

§ 5. We shall now formulate analytically a condition for the Hill stability of two bodies, which we denote here by P_1 and P_2, assuming $\mu_1 \geq \mu_2$. We have seen that, for $\rho > \xi_2$, $S_3(\rho) \geq s$, if by ρ one understands those values which are realized in reality. Let $s > S_3(L_3) = S_{3\,\min}$, so that the equation $S_3(\rho) = s$ has two roots ρ' and ρ'', with $\xi_2 < \rho' < \xi(L_3) < \rho''$. Then, if at the initial instant $\rho \geq \rho''$, this inequality will be satisfied at every instant of time. Since $\rho = O_{12}P_3/P_1P_2 \geq \rho'' > \xi_2 \geq |\xi_1|$, it follows, in particular, that the motion of the bodies P_1 and P_2 is Hill stable. This theorem cannot be improved if only the parameter s and the initial value of the quantity ρ are used.

Consider, for example, the case $\mu_1 = \mu_2 = \mu_3 = 1/3$. Since $S_{3\,\min} = 25/486 \simeq 0.05144$, the Hill stability of a definite pair of bodies in such a system will already occur for $s > 0.05144$. Application of G. A. Merman' s theorems ([4]) gives the following. Merman' s original theorem guarantees Hill stability only for $s > 8/\sqrt{3} \simeq 4.618$; Theorem 1 for $s > 81\sqrt{2}/16 \simeq 7.160$; Theorem 1 bis for $s > (57 + 8\sqrt{41})/36 \simeq 3.006$.

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Note: Figure translations are in progress. See original paper for figures.