ON REGULAR DIRICHLET DECOMPOSITIONS FOR THE SECOND TRICLINIC GROUP

1968

SovietRxiv

Abstract

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UDC 513.348

MATHEMATICS

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(Presented by Academician I. M. Vinogradov on 1 VII 1968)

The question of regular decompositions of space has long been considered, but the first nontrivial results in this direction were obtained only in 1961 in the work [(1)], in which the finiteness of the number of combinatorially distinct such decompositions for Euclidean spaces of any given number of dimensions was proved and an algorithm was proposed for finding the most important of them –the Dirichlet decompositions. But already for three-dimensional space this algorithm is hardly practicable.

In the present note a method is proposed for finding all kinds of Dirichlet decompositions of three-dimensional Euclidean space for the second triclinic group T_2 (in crystallographers' notation $P\bar{1}$).

§ 1. **The method of "white" faces.** As is known, if one takes a point O, not multiple with respect to T_2, and acts on it by all elements of this group, then one obtains a regular system of points representing a bilattice, consisting of two identical lattices Γ and Γ' parallel to one another. The metric of the lattice Γ and the displacement vector a of the lattice Γ' with respect to Γ may be arbitrary, provided only that Γ and Γ' do not coincide.

We shall construct the Dirichlet stereohedron S of the point O of the bilattice in the following way. Let the point O belong to the lattice Γ. Construct the Dirichlet parallelohedron D of the point O of the lattice Γ and all Dirichlet parallelohedra D' of the lattice Γ'. The parallelohedron D intersects certain parallelohedra D' of the decomposition $\{D'\}$ in centrally symmetric pieces; the decomposition of the parallelohedron D into such pieces will be called a **development**. Join the point O by segments to the centers of the parallelohedra D' intersecting the parallelohedron D in three-dimensional pieces, and draw the Voronoi planes p through the midpoints of these segments, i.e., through the centers of the pieces. The planes p divide the corresponding pieces in half. The halves of the pieces lying, with respect to the planes p, on the same side as the point O obviously constitute the required stereohedron S.

The faces of the stereohedron S that are parts of faces of the parallelohedron D will be called **black**. The black faces of the stereohedron are pairwise equal and parallel. Those faces of the stereohedron that lie in the planes p will be called **white**. The white faces of the stereohedron are centrally symmetric.

The adjacency symbol of the decomposition $\{S\}$ consists in the fact that along black faces the stereohedra are applied to one another by parallel translations, and along white ones—by reflections in their centers.

Proceeding from what has been said above, all kinds of decompositions $\{S\}$ can be found in two stages: first find all combinatorial types of developments, and then find all possible combinatorially distinct arrangements of white faces with respect to the developments.

§ 2. **On combinatorial types of developments.** A development will be called **general** if its combinatorial type does not change under an arbitrary sufficiently small change of the metric of the bilattice.

Let us call the totality of faces of the parallelohedra D' of the decomposition $\{D'\}$ a **net** and denote it by $[D']$. Let, further, A be one of the vertices of the parallelo-

the parallelohedron D, and let $A'_1, A'_2, \dots, A'_\nu$ be a complete set of vertices of the lattice $[D']$, pairwise inequivalent with respect to the subgroup of parallel translations T of the group T_2, where $\overline{AA'_i} = \mathbf{a}$. Translating the lattice $[D']$ parallel to itself so that successively the points $A'_1, A'_2, \dots, A'_\nu$ coincide with the point A, we obtain ν lattices, whose totality we shall call a **multilattice**. The multilattice divides the parallelohedron D into cells. The interiors of the cells, of their two-dimensional faces and edges, and also the vertices, are respectively 3-dimensional, 2-dimensional, 1-dimensional, and 0-dimensional **regions of a partition**, within which, as the point A'_1 is moved, the combinatorial type of the partition does not change.

If g, h, k, l, m, n are the reduced Selling parameters of the lattice Γ, then it can be shown that, under inequalities of the type

$$n < g, \qquad l < g, \qquad k < g, \qquad g < h, \qquad m < h, \qquad h < 0$$

the multilattice is general, and when at least one of these inequalities is turned into an equality it is special. By going through all combinatorially different multilattices and taking one parallelohedron associated with each of them, one can find all combinatorial types of partitions. It turned out that there are in all 4 general and 98 special combinatorial types of partitions, not counting 5 trivial ones, when Γ and Γ' coincide.

Remark. As a by-product of the investigation, a direct geometric derivation was obtained of all possible 4-dimensional parallelohedra, if one regards as known the theorem proved by B. N. Delone that every 4-dimensional parallelohedron has at least one closed belt of edges. In his well-known memoir

([2]), B. N. Delone had to proceed by a roundabout route, namely: first to prove Voronoi' s proposition that every parallelohedron is an affine image of a Dirichlet parallelohedron. In particular, in B. N. Delone' s table one 52nd 4-dimensional parallelohedron is missing.

To each birette we assign a point $M(g,h,k,l,m,n,x,y,z)$ of the 9-dimensional space of parameters (the coordinates of the vector **a** are denoted by x,y,z). We construct the partition for a fixed birette and continuously move the corresponding point M in the 9-dimensional space in such a way that the partition does not change combinatorially. The totality of all points M to which one can thus pass from the given one will be called the **region of the type of the given partition** (if the lattice parameters are fixed, we obtain the previously described regions of partitions). It turned out that each region of a partition type is a convex polyhedron.

§ 3. **Finding all sorts of Dirichlet partitions for the group T_2.** A partition $\{S\}$ of space into Dirichlet stereohedra S for the group T_2 will be called **general** if it does not change combinatorially under an arbitrary sufficiently small change of the metric of the lattice Γ and of the vector **a**. Otherwise the partition will be called **special**.

Two partitions $\{S\}$, in the terminology of B. N. Delone, are of one **sort** if they are combinatorially identical and the group under consideration, in the present case the group T_2, acts on them in the same way.

We shall look for all sorts of Dirichlet partitions $\{S\}$ corresponding to points M of some region of a partition type. Construct the Dirichlet stereohedron S for some fixed point M of this region (we shall call it the **initial** one). For a fixed point M this can always be done. Take some vertex of the partition and draw from it the diameters (i.e., centrally symmetric chords) of all those pieces to which it belongs. For each end of a diameter we repeat the same procedure, and so on. The resulting collection of diameters will be called a **lightning**. We shall call a lightning **critical** with respect to the stereohedron S if its vertices satisfy one of two requirements: 1) each vertex of the lightning in every piece of the partition to which it belongs is the end of at least one edge that intersects the white face lying inside this piece, in the

interior point of it, or 2) all vertices of the lightning lie on white faces, but they are not all equivalent under the group T.

If the point M is moved continuously in the given region of the type of unfolding, then the stereohedron S changes continuously; moreover, the sort of the partition $\{S\}$ can change for the first time only at the moment when at least one critical lightning is triggered, i.e., when the vertices of at least one critical lightning which did not lie on white faces fall onto them, or when the vertices of at least one critical lightning which did lie on white faces leave them. But not every critical lightning can be triggered first.

Fig. 1

Fig. 1

Figure 1: Fig. 1

The proposed method consists of the following. We seek all critical lightnings with respect to the initial stereohedron S. Acting purely combinatorially, we construct the so-called *conditional stereohedra*, into which the stereohedron S would be transformed if first one or another critical lightning, or some combination of them, were triggered. Those conditional stereohedra which plainly cannot be realized as convex polyhedra whose white faces are centrally symmetric and whose black faces are pairwise equal and parallel are discarded. For each of the remaining conditional stereohedra, in turn, as for the initial one, we find the critical lightnings and find the further remaining conditional stereohedra, etc.

By virtue of the finiteness of the number of combinatorially different positions of the white faces with respect to the unfolding, this process will end, and we shall obtain a complete set of the remaining conditional stereohedra differing from one another in the combinatorial position of the white faces with respect to the unfolding. Any Dirichlet stereohedron S corresponding to any point M of the given region of the type of unfolding is, evidently, combinatorially identical with some conditional stereohedron of the complete set. In the actual verification of the method it turned out that all conditional stereohedra of any such complete set are realized as Dirichlet stereohedra for certain choices of the lattice metric.

Let us show the operation of this method by the example of deriving all general sorts of partitions $\{S\}$ for T_2. For this it is sufficient to take only general unfoldings (in this case D is a 14-hedron) and to construct conditional stereohedra when lightnings are triggered only one at a time. Let us take the first of them, which is the projection of the cap of a 4-dimensional parallelohedron of type I along an edge of the zone. The vertices of this unfolding are situated on 12 lightnings. The 14-hedron has hexagonal faces on which there is exactly one representative vertex from each lightning; therefore, in order to determine the combinatorial position of the white faces with respect to all vertices of the unfolding, it is sufficient to know this only with respect to the vertices of the unfolding situated on such a face.

In the figures of Fig. 1 the trace of the net $[D']$ on this face is shown by a dashed line, and the hatched polygons are the black faces of the conditional stereohedra which are part of this face. We take as the initial one such a stereohedron S_a (Fig. 1a), which is obtained when only that zone of the 14-hedron which does not contain this uncut face is sufficiently elongated, and for such a vector **a** for which both the 4 upper and the 4 lower, with respect to ⋯

with respect to this zone its pieces are also strongly elongated. With respect to S_F, only the zippers 1 and 2 are critical. When 1 is triggered (analogously, 2), we obtain the conditional stereohedron S_6 (Fig. 16). With respect to S_6, only the zippers 1, 2, 3, 4 are critical. When 3 is triggered (analogously, 4), we obtain S_8, and when 2 is triggered, S_∂. With respect to S_8, only the zippers 1, 2, 3, 4, 5, 6 are critical, but we discard the conditional stereohedra obtained when 1, 4, and 6 are triggered, since there can be no such convex stereohedra. If one draws the figures at each new triggering (discarding the conditional stereohedra that are manifestly unrealizable as convex ones), then there will be 22 of them, but by virtue of the combinatorial symmetry of the net they all reduce to the 5 that have been drawn. If the vertices of the 14-hedron are situated on three concentric spheres of different radii and vertex 5 (Fig. 12) lies on the sphere of smallest radius, then it is easy to choose the vector **a** so that the Dirichlet stereohedron S_2 is obtained; then, by stretching one of the zones and changing the vector **a**, one can obtain the Dirichlet stereohedra S_8 and S_6. The Dirichlet stereohedron S_∂ is obtained, for example, when the net is metrically most symmetric. The same 5 types are obtained for the second net. For the third net we obtain 9 types. For the fourth we also obtain 9 types, but 8 of them are the same as for the third net.

In summary we obtain: there exist altogether 15 general types of Dirichlet partitions $\{S\}$ for the group T_2 (among them 5 partitions into 20-hedra, 5 into 18-hedra, 2 into 16-hedra, and 3 into 14-hedra).

Investigating the limiting cases in the same way, we obtain another 165 special types of Dirichlet partitions. Almost all these partitions are also combinatorially distinct (among the 15 general ones, for example, only 2 partitions of different types are combinatorially identical).

In conclusion I express my sincere gratitude to B. N. Delone for his comprehensive assistance in carrying out this work.

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Received
20 VI 1968

CITED LITERATURE

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2. B. N. Delone, *UMN*, vol. 3, 16 (1937); vol. 4, 102 (1938).

Note: Figure translations are in progress. See original paper for figures.