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Abstract

Full Text

MATHEMATICS

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ON CONDITIONS FOR THE EXISTENCE OF A SOLUTION OF ONE PROBLEM OF OPTIMAL CONTROL

(Presented by Academician L. S. Pontryagin on 10 VI 1968)

Consider a system of linear differential equations with constant coefficients (in vector notation)

$$\frac{dx}{dt} = Ax + \sum_{s=1}^{r} u_s v^s(t) \quad (x = x(t) \subset R^n; \quad u_s \subset R^n,\ s = 1, 2, \dots, r)\,. \qquad (1)$$

Here $v^s(t)$, $s = 1, 2, \dots, r$, are functions piecewise continuous on any finite interval in t and satisfying the following additional condition: for arbitrary $t = t_1$, the point

$$u(t_1) = \sum_{s=1}^{r} u_s v^s(t_1)$$

belongs to some previously chosen domain $\overline{U}$ of the subspace $R^r \subset R^n$, defined by linearly independent vectors u_s, $s = 1, 2, \dots, r$. We assume the control domain $\overline{U}$ to be closed and to contain any sphere of sufficiently small radius with center at the origin.

In the usual problem of optimal speed ([1,2]), it is required, for a point $x_0 \in R^n$, to indicate an admissible control $v(t) = \{v^s(t)\}$, $s = 1, 2, \dots, r$, which transfers it along a trajectory of system (1) to the origin in the minimum time.

Below we consider a condition somewhat modified in comparison with the usual one. We shall say that the condition of general position of the control vectors u_s, $s = 1, 2, \dots, r$, is satisfied if one can indicate such a set g of n pairs of natural numbers

$$g = \{(m_i, s_i)\}, \quad i = 1, 2, \dots, n,$$

that the determinant composed of column vectors of the form $A^{m_i} u_{s_i}$, completely specified by the set g, exists and is not equal to zero. It can be shown that this condition is invariant with respect to nonsingular linear transformations of the space R^n or of the subspace R^r.

Let us recall the well-known definition of controllable systems of differential equations. System (1) is called controllable if there exists a controllability domain $Q \subset R^n$, open and containing the origin, from each point of which one can reach the origin in finite time, moving along a trajectory of system (1) under some admissible control $v(t)$ ([2]). It can be shown that controllability of system (1) is determined by the choice of the space R^r and depends neither on the choice of the domain $\overline{U}$, nor on the choice of the basis vectors u_s, $s = 1, 2, \dots, r$, in this space. The controllability domain, of course, depends on the choice of the domain $\overline{U}$.

Theorem. *In order that the system of differential equations (1) be controllable, it is necessary and sufficient that the condition of general position of the control vectors u_s, $s = 1, 2, \dots, r$, be satisfied,*

$$\left| A^{m_i} u_{s_i} \right|_{(g)} \neq 0. \tag{2}$$

Let us explain the proof of the theorem.

Sufficiency of the condition. Let us write the solution of system (1) for some $t = h$:

$$x(h) = e^{Ah} x_0 + e^{Ah} \int_0^h e^{-A\tau} u(\tau)\, d\tau.$$

Requiring that, for an arbitrary $t = h$, the solution fall at the origin of the coordinates, we obtain the equation

$$-\int_0^h e^{-A\tau} u(\tau)\, d\tau = x_0. \tag{3}$$

Divide the interval $[0, h]$ into n parts by as yet undetermined points t_i, $i = 1, 2, \dots, n$, and put $u(\tau) = a^i u_s$ in each of the corresponding partial intervals. The numbers a^i, $i = 1, 2, \dots, n$, will for the time being be left undetermined, while the numbers s_i, $i = 1, 2, \dots, n$, will be taken from the set g. Writing in (3) the matrix function $e^{-A\tau}$ in the form of a power series and integrating on each of the partial intervals, we obtain a linear algebraic system for finding the numbers a^i, $i = 1, 2, \dots, n$,

$$\sum_{i=1}^{n} \varphi_i(A) a^i u_{s_i} = x_0.$$

Let us write the determinant of this system:

$$\det \|\varphi_i(A)u_{s_i}\| = \sum_{\{g\}} \psi_g(t_1, \dots, t_n).$$

By condition (2), at least one of the determinants d_g occurring in the right-hand side is different from zero. The polynomials ψ_g cannot vanish identically and will not be linearly dependent for different g. Therefore one can choose numbers t_i, $i = 1, 2, \dots, n$, from the interval $[0, h]$ such that the determinant of the indicated system does not vanish. Then, for any point x_0 of some arbitrary open domain Q, $Q \ni 0$, having sufficiently small diameter, numbers a^i will be found such that the vectors $a^i u_{s_i}$, $i = 1, 2, \dots, n$, will belong to the control region $\overline{U}$. System (1) is controllable.

Necessity of the condition. By a linear transformation of system (1), reduce the matrix A to (lower) Jordan form. Introduce the notation: λ_j, $j = 1, 2, \dots, \sigma$, are the distinct eigenvalues of the matrix A; σ is their number; q_j is the number of elementary blocks of the form

$$B_{jp} = B_{jp}(\lambda_j) = \begin{pmatrix} \lambda_j & 0 & \dots & 0 & 0 \\ 1 & \lambda_j & \dots & 0 & 0 \\ 0 & 0 & \dots & 1 & \lambda_j \end{pmatrix},$$

containing the eigenvalue λ_j; l_{jp} is the number of rows (columns) in these blocks, $p = 1, 2, \dots, q_j$, $j = 1, 2, \dots, \sigma$. Combine the blocks into arrays (of the first kind) $M_j^* = (B_{jp})$, $p = 1, 2, \dots, q_j$, corresponding to the eigenvalue λ_j, $j = 1, 2, \dots, \sigma$. Within each array arrange the blocks according to the relation $l_{jp} \geq l_{j,p+1}$, $p = 1, 2, \dots, q_j - 1$. Arrange the sequence of arrays M_j^*, $j = 1, 2, \dots, \sigma$, by means of the numbers $q_j; l_{j1}, \dots, l_{jq_j}$: let $q_j \geq q_{j+1}$ and $l_{j1} \geq l_{j+1,1}$, if $q_j = q_{j+1}$, etc. This ordering determines the order and numbering of the components of the vectors x and u_s, $s = 1, 2, \dots, r$, of system (1).

From the components of the vectors u_s, $s = 1, 2, \dots, r$, form the matrices

$$U_j = \|u_s^{m_{jp}+1}\|, \qquad p = 1, 2, \dots, q_j; \quad s = 1, 2, \dots, r,$$

for all $j = 1, 2, \dots, \sigma$.

Here, the letters m_{jp}, $p = 1, 2, \dots, q_j$; $j = 1, 2, \dots, \sigma$, denote the numbers of the rows preceding the cell B_{jp} in the matrix $\hat{A}$. Let us find the numbers

$$r_j = \operatorname{rank} U_j, \qquad j = 1, 2, \dots, \sigma. \tag{4}$$

In what follows we shall study the conditions

$$r_j = q_j, \qquad j = 1, 2, \dots, \sigma. \tag{5}$$

In addition to the arrays $M_j^*,\ j=1,2,\dots,\sigma$, we shall need arrays (of the second kind) $M_p^{**}=(B_{jp}),\ j=1,2,\dots,\sigma_p$, constructed for all $p=1,2,\dots,q_1$. Here the numbers σ_p are determined from the conditions $q_{\sigma_p}\geq p,\ q_{\sigma_p+1}<p$. We shall denote by the letters h_p the numbers of rows (columns) in the arrays $M_p^{**},\ p=1,2,\dots,q_1$.

Let us prove that conditions (5) are necessary conditions for controllability of system (1). Suppose that for some one $j=1,2,\dots,\sigma$ we obtain the inequality $r_j<q_j$. From system (1) choose all equations with numbers $m_{jp}+1,\ p=1,2,\dots,q_j$. They form a subsystem (in vector form)

$$\frac{d\hat{x}}{dt}=\lambda_j\hat{x}+\sum_{s=1}^{r}\hat{u}_s v^s(t).$$

Construct a nonsingular transformation $\hat{x}=P\hat{y}$ of the corresponding subspace $R^{q_j}\subset R^n$. As the first r_j columns of the matrix P take those r_j linearly independent vectors which are found among the vectors $\hat{u}_s,\ s=1,2,\dots,r$, by the definition (4) of the numbers $r_j,\ j=1,2,\dots,\sigma$. Then the indicated subsystem in the new coordinates will have unit vectors as control vectors. The number of these vectors is less than the dimension of the space R^{q_j}, $r_j<q_j$. Therefore the last q_j-r_j equations will not contain controls and will have the form

$$dy^{r_j+i}/dt=\lambda_j y^{r_j+i},\qquad i=1,2,\dots,q_j-r_j.$$

This system has the trivial solution $u^{r_j+i}=0,\ i=1,2,\dots,q_j-r_j$. In the space R^n it forms a linear manifold L containing the origin. The solution of system (1), determined by an initial point $x_0\notin L$, does not enter L in finite time under any admissible control. The controllability domain cannot be constructed, and system (1) is uncontrollable.

We proceed to the proof of the assertion that fulfillment of condition (5) implies fulfillment of condition (2). If $r_j=q_j,\ j=1,2,\dots,\sigma$, then the numbers $s_{jp},\ p=1,2,\dots,q_j$, can be so chosen from the numbers $s=1,2,\dots,r$ that the conditions

$$\tilde{\Delta}_{jp}=\left|u_{s_{jp}}^{m_{jk}+1}\right|_{p,k=1,2,\dots,q}\neq 0 \tag{6}$$

will be satisfied for all $q=1,2,\dots,q_j$ and $j=1,2,\dots,\sigma$.

These sets of numbers $s_{jp},\ p=1,2,\dots,q_j$, are different for different $j=1,2,\dots,\sigma$. We shall replace the vectors $u_s,\ s=1,2,\dots,r$, by new vectors $v_p,\ p=1,2,\dots,r$, taken in the form of linear combinations of the former vectors. We require that, for the components of the new vectors, the conditions

$$\Delta_{jq}=\left|v_p^{m_{jk}+1}\right|_{p,k=1,2,\dots,q}\neq 0 \tag{7}$$

be satisfied for all $q = 1, 2, \dots, q_j$ and $j = 1, 2, \dots, \sigma$ simultaneously. Put $v_{1,p} = u_{s_{1p}},\ p = 1, 2, \dots, q_1$. Conditions (7) are satisfied for $j = 1$ and $q = 1, 2, \dots, q_1$.

Suppose that the vectors $v_{n,p},\ p = 1, 2, \dots, q_1$, have been constructed and that conditions (7) are satisfied for $j = 1, 2, \dots, h$ and $q = 1, 2, \dots, q_j$. Put $v_{n+1,p} = v_{n,p}+$

$+\alpha^{n+1} u_{s\,n+1,p},\ 1 \leq p \leq q_{n+1},\ v_{n+1,p} = v_{h,p},\ q_{n+1} < p \leq q_1$, where α^{n+1} is an undetermined coefficient. Then, with the aid of conditions (6), one can choose the number a^{h+1} so that condition (7) will be satisfied for $j = 1, \dots, h+1$ and $q = 1, \dots, q_j$. For $h = q_1$ put $v_p = v_{q_1,p}$, $p = 1, 2, \dots, q_1$. For these vectors all conditions (7) are satisfied. It is not difficult to complete them to a basis in R^r, if $q_1 < r$.

We shall first introduce the construction of the determinant d_q (2) by means of the arrays M_p^{**}, $p = 1, 2, \dots, q_1$. To each such array M_p^{**} we assign a vector v_p, $p = 1, 2, \dots, q_1$. Then to each column of this array we assign a vector of the form $A^{k_p} v_p$ with increasing exponent $k_p = 0, 1, \dots, h_p - 1$, where h_p is the number of columns in M_p^{**}. Thus to each column of the matrix A there will correspond one vector of the form $A^{k_p} v_p$. From these vectors we form a determinant, arranging them according to the ordering of the corresponding columns of the matrix A. The order in which the vectors $A^{k_p} v_p$ occur in the determinant obtained will correspond to the distribution of the columns of the matrix A among the arrays M_j^*, and not among the arrays M_p^{**}. We denote the determinant with this ordering of the columns by d,

$$d = \left| A^{k_p} v_p \right|. \tag{8}$$

Lemma. To each array M_j^*, $j = 1, 2, \dots, \sigma$, assign the number

$$d_j^* = \prod_{q=1}^{q_j} \Delta_{jq}^{\,l_{jq} - l_{j,q+1}}$$

(where $l_{j,q_j+1} = 0$ is put), and to each array M_p^{**}, $p = 1, 2, \dots, q_1$, the number

$$d_p^{**} = \prod_{j',\,j=1,\ j'>j}^{\sigma_p} (\lambda_{j'} - \lambda_j)^{-l_{j'p}\, l_{jp}}.$$

Then the determinant d (8) is computed by the formula

$$d = \prod_{j=1}^{\sigma} d_j^* \prod_{p=1}^{q_1} d_p^{**}. \tag{9}$$

In proving this lemma, we replace columns of the form $A^{k_p} v_p$ by columns of the form

$$(A-\lambda_{jE})^{s-1}\prod_{i=1}^{j-1}(A-\lambda_{iE})^{l_{ip}}v_p, \qquad s=1,2,\ldots,l_{jp},$$

for each column with number s from the cell B_{jp}. This replacement does not change the value of the determinant d. Then all elements standing to the right and above the arrays M_j^* are annihilated. After this we compute the determinants corresponding to the arrays M_j^*.

Using the lemma, it is easy to show that the determinant d (8) is different from zero. The theorem is proved.

The main theorem can also be formulated in the following form:

If the matrix A is reduced to Jordan form, then a necessary and sufficient condition for the controllability of system (1) is the fulfillment of conditions (5).

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2. V. G. Boltyanskii, *Mathematical Methods of Optimal Control*, "Nauka," 1966.

Note: Figure translations are in progress. See original paper for figures.