EMBEDDING THEOREMS FOR WEIGHTED FUNCTION SPACES

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Abstract

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EMBEDDING THEOREMS FOR WEIGHTED FUNCTION SPACES

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In the present paper we consider embedding theorems for the weighted function spaces $W^r_{p,\alpha}(\overset{+}{E}{}^n)$, $L^r_{p,\alpha}(\overset{+}{E}{}^n)$ into a space with mixed norm. E^n is the set of points $\{x_1, x_2, \ldots, x_n\}$ of the n-dimensional Euclidean space E^n for which $x_n > 0$.

Let $f(\bar{x})$ be a smooth function defined in the space E^n. The numbers α, r_i $(i = 1, \ldots, n)$, and p are real and satisfy the conditions $\alpha > -1$, $0 < r_i < 1$, $1 \leq p \leq \infty$. We shall say that $f \in L^{r_i}_{p,\alpha}(\overset{+}{E}{}^n)$ if

$$\|f\|_{L^{r_i}_{p,\alpha}(\overset{+}{E}{}^n)} = \left(\int_0^\infty \frac{dt}{t^{1+pr_i}} \int_{\overset{+}{E}{}^n} x_n^\alpha \left| \Delta_i(t) f(\mathbf{x}) \right| d\mathbf{x} \right)^{1/p} < \infty,$$

$$\|f\|_{L^{r}_{p,\alpha}(\overset{+}{E}{}^n)} = \sum_{i=1}^n \|f\|_{L^{r_i}_{p,\alpha}(\overset{+}{E}{}^n)},$$

$$\|f\|_{W^{r}_{p,\alpha}(\overset{+}{E}{}^n)} = \sum_{i=1}^n \|f\|_{L^{r_i}_{p,\alpha}(\overset{+}{E}{}^n)} + \|f\|_{L_p(\overset{+}{E}{}^n)}.$$

The closures of smooth finite functions in the corresponding norms will be denoted by $W^r_{p,\alpha}(\overset{+}{E}{}^n)$ and $L^r_{p,\alpha}(\overset{+}{E}{}^n)$. We shall say that $f(\mathbf{x}) \in L^{\rho_s}(p, p_1, p_2, \ldots, p_m)(E^m)$ if

$$\|f\|_{L^{\rho_s}_{(p,p_1,p_2,\ldots,p_m)}(E^m)} = \left(\int_0^\infty \frac{dt}{t^{1+\rho_s p}} \|\Delta_s(t) f(\mathbf{x})\|^p_{L(p_1,p_2,\ldots,p_m)(E^m)} \right)^{1/p} < \infty,$$

$$\|f\|_{\vec{L}^{\rho}_{(p,p_1,p_2,\ldots,p_m)}(E^m)} = \sum_{s=1}^m \|f\|_{L^{\rho_s}_{(p,p_1,p_2,\ldots,p_m)}(E^m)},$$

$$\|f\|_{\vec{W}^{\rho}_{(p,p_1,p_2,\dots,p_m)}(E^m)} = \|f\|_{L(p_1,\dots,p_m)(E^m)} + \|f\|_{\vec{L}^{\rho}_{(p,p_1,\dots,p_m)}(E^m)},$$

where $0 < \rho_s < 1$ $(s = 1, \dots, m)$,

$$\|f\|_{L(p_1,p_2,\dots,p_m)(E^m)} = \left(\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\cdots\left(\int_{-\infty}^{\infty}|f|^{p_1}\,dx_1\right)^{p_2/p_1}\cdots\right)^{p_m/p_{m-1}}dx_m\right)^{1/p_m} < \infty.$$

We formulate the results obtained in the form of embedding theorems.

Theorem 1. If $f(\bar{x}) \in W^r_{p,\alpha}(\overset{+}{E}{}^n)$ and the conditions

$$1 < p \le p_1 \le \dots \le p_m < \infty, \quad 0 < m \le n-1, \quad \alpha > -1,$$

$$0 < r_i < 1 \quad (i = 1, \dots, n),$$

are satisfied,

$$\varepsilon = 1 + \sum_{i=1}^{m}\frac{1}{p_i r_i} - \frac{1}{p}\sum_{i=1}^{n}\frac{1}{r_i} - \frac{\alpha}{p r_n} > 0,$$

then, for $x_n = 0$, $f \in L_{(p_1,p_2,\dots,p_m)}(E^m)$, and the inequality

$$\|f\|_{L_{(p_1,p_2,\dots,p_m)}(E^m)} \leqslant c\|f\|_{W^r_{p,\alpha}(E^n+)}$$

holds.

Theorem 2. If $f(x) \in L^r_{p,\alpha}(E^n+)$ and the conditions

$$1 < p \leqslant p_1 \leqslant \dots \leqslant p_m < \infty, \quad 0 < m \leqslant n-1, \quad \alpha > -1,$$

$$0 < r_i < 1 \quad (i = 1, \dots, n),$$

$$\varepsilon = 1 + \sum_{i=1}^{m}\frac{1}{p_i r_i} - \frac{1}{p}\sum_{i=1}^{n}\frac{1}{r_i} - \frac{\alpha}{p r_n} > 0,$$

are satisfied, then, for $x_n = 0$, $f \in L_{(p_1,p_2,\dots,p_m)}(E^m)$, and the inequality

$$\|f\|_{L_{(p_1,p_2,\dots,p_m)}(E^m)} \leqslant c\|f\|_{L^r_{p,\alpha}(E^n+)}$$

holds.

Theorem 3. If $f(x) \in W^r_{p,\alpha}(E^n+)$ and the conditions

$$1 < p \leqslant p_1 \leqslant \dots \leqslant p_m < \infty, \quad 0 < m \leqslant n-1, \quad \alpha > -1,$$

$$0 < r_i < 1 \quad (i = 1, \dots, n),$$

$$0 < \rho_s \leqslant \varepsilon r_s \quad (s = 1, \dots, m), \qquad \varepsilon = 1 + \sum_{i=1}^{m} \frac{1}{p_i r_i} - \frac{1}{p}\sum_{i=1}^{n}\frac{1}{r_i} - \frac{\alpha}{p r_n} > 0,$$

are satisfied, then, for $x_n = 0$, $f \in L^{\vec{\rho}}_{(p,p_1,\dots,p_m)}(E^m)$, and the inequality

$$\|f\|_{L^{\vec{\rho}}_{(p,p_1,\dots,p_m)}(E^m)} \leqslant c\|f\|_{W^r_{p,\alpha}(E^n+)}$$

holds.

Theorem 4. If $f(x) \in W^r_{p,\alpha}(E^n+)$ and the conditions

$$1 < p \leqslant p_1 \leqslant \dots \leqslant p_m < \infty, \quad 0 < m \leqslant n-1, \quad \alpha > -1,$$

$$0 < r_i < 1 \quad (i = 1, \dots, n),$$

$$0 < \rho_s \leqslant \varepsilon r_s \quad (s = 1, \dots, m), \qquad \varepsilon = 1 + \sum_{i=1}^{m} \frac{1}{r_i p_i} - \frac{1}{p}\sum_{i=1}^{n}\frac{1}{r_i} - \frac{\alpha}{p r_n} > 0,$$

are satisfied, then, for $x_n = 0$, $f \in W^{\vec{\rho}}_{(p,p_1,\dots,p_m)}(E^m)$, and the inequality

$$\|f\|_{W^{\vec{\rho}}_{(p,p_1,\dots,p_m)}(E^m)} \leqslant c\|f\|_{W^r_{p,\alpha}(E^n+)}$$

holds.

In proving these theorems, the integral representation of V. P. Il' in ([2]) is used.

We shall outline the proof of one of the stated theorems, for example, Theorem 3.

We have

$$\|f\|_{L^{\rho_s}_{(p,p_1,\dots,p_m)}(E^m)} = \left(\int_0^\infty \frac{dt}{t^{1-\rho_s p}} \|\Delta_s(t)f(x)\|^p_{L_{(p_1,p_2,\dots,p_m)}(E^m)}\right)^{1/p}; \tag{1}$$

we represent the function $f(x)$ in the form

$$f(x) = \sum_{i=1}^{?} f_i, \tag{2}$$

where

$$f_1(\bar{x}, h) = \frac{c}{h^\omega} \int_0^{h^{\sum_1^n \sigma_j}} f(\bar{x} + \bar{y}) \Pi(\bar{y}, h)\, d\bar{y},$$

$$f_2(\bar{x}, t, h) = -c \sum_{i=1}^n \int_0^{t^{1/\sigma_s}} \frac{dv}{v^{1+\omega}} \int_0^{v^{\sum_1^n \sigma_j}} d\bar{y} \int_0^{v^{\sigma_i - y_i}} \Delta_i(\tau) f(\bar{x} + \bar{y}) R_i(\bar{y}, \tau, v)\, d\tau,$$

$$f_3(x, t, h) = -c \sum_{i=1}^n \int_{t^{1/\sigma_s}}^h \frac{dv}{v^{1+\omega}} \int_0^{v^{\sum_1^n \sigma_j}} d\bar{y} \int_0^{v^{\sigma_i - y_i}} \Delta_i(\tau) f(\bar{x} + \bar{y}) R_i(\bar{y}, \tau, v)\, d\tau,$$

where $\sigma_j = 1/r_j$.

Taking equality (2) into account, we have

$$\|\Delta_s(t) f(x)\|_{L_{(p_1,p_2,\dots,p_m)}(E^m)} \le \sum_{i=1}^3 \|\Delta_s(t) f_i\|_{L_{(p_1,p_2,\dots,p_m)}(E^m)}. \tag{3}$$

The following estimates hold:

$$\|\Delta_s(t) f_1(\bar{x}h)\|_{L_{(p)}(E^m)} \le c t h^{-\frac{1}{p}\sum_1^n \sigma_j + \sum_1^m \frac{\sigma_j}{p_j} - \sigma_s} \|f\|_{L_p^+(E^n)}; \tag{4}$$

$$\|\Delta_s(t) f_2(\bar{x}, t, h)\|_{L_{(p)}(E^m)} \le c \sum_{i=1}^n \left\| \int_0^{t^{1/\sigma_s}} \frac{dv}{v^{1+\lambda_i - \varepsilon + \gamma\sigma_i - \alpha/pr_n}} \times \right.$$

$$\left. \times \int_0^{v^{\sigma_i}} \tau^\gamma\, d\tau \int_x^{x+v^{\sum_1^n \sigma_j}} \frac{|\Delta_i(\tau) f(\bar{y})|}{\tau^{1/p + r_i}}\, d\bar{y} \right\|_{L_{(p)}(E^m)}; \tag{5}$$

$$\|\Delta_s(t) f_3(\bar{x}, t, h)\|_{L_{(p)}(E^m)} \leq ct \sum_{i=1}^{n} \left\| \int_{t^{1/\sigma_s}}^{h} \frac{dv}{v^{1+\lambda_i - \varepsilon + \gamma\sigma_i + \sigma_s - \alpha/pr_n}} \times \right.$$

$$\left. \times \int_0^{v^{\sigma_i}} \tau^{\gamma} \, d\tau \int_x^{x+v^{\sum_1^n \sigma_j}} \frac{|\Delta_i(\tau) f(\bar{y})|}{\tau^{1/p+r_i}} \, d\bar{y} \right\|_{L_{(p)}(E^m)}. \tag{6}$$

We have

$$\left(\int_0^{\infty} \frac{dt}{t^{1+\rho_s p}} \|\Delta_s(t) f(\bar{x})\|^p_{L_{(p)}(E^m)} \right)^{1/p} \leq$$

$$\leq \left(\int_0^{h^{\sigma_s}} \frac{dt}{t^{1+\rho_s p}} \|\Delta_s(t) f(\bar{x})\|^p_{L_{(p)}(E^m)} \right)^{1/p} + \left(\int_{h^{\sigma_s}}^{\infty} \frac{dt}{t^{1+\rho_s p}} \|\Delta_s(t) f(\bar{x})\|^p_{L_{(p)}(E^m)} \right)^{1/p} \leq$$

$$\leq \sum_{i=1}^{3} \left(\int_0^{h^{\sigma_s}} \frac{dt}{t^{1+\rho_s p}} \|\Delta_s(t) f_i\|^p_{L_{(p)}(E^m)} \right)^{1/p} + \left(\int_{h^{\sigma_s}}^{\infty} \frac{dt}{t^{1+\rho_s p}} \|\Delta_s(t) f(\bar{x})\|^p_{L_{(p)}(E^m)} \right)^{1/p} =$$

$$= \sum_{i=1}^{3} A_i + B. \tag{7}$$

If we take into account (4), (5), and (6), then after a series of estimates we obtain

$$A_1 = \left(\int_0^{h^{\sigma_s}} \frac{dt}{t^{1+\rho_s p}} \|\Delta_s(t) f_1\|^p_{L(p)(E^m)} \right)^{1/p} \ll ch^{-(\delta+\rho_s\sigma_s)} \|f\|_{L_p(\overset{+}{E^n})},$$

$$A_2, \; A_3 \ll ch^{\varepsilon - \rho_s\sigma_s} \|f\|_{L^r_{p,\alpha}(\overset{+}{E^n})}.$$

Consequently,

$$\sum_{i=1}^{3} A_i \ll c \left(h^{-(\delta+\rho_s\sigma_s)} \|f\|_{L_p(\overset{+}{E^n})} + h^{\varepsilon-\rho_s\sigma_s} \|f\|_{L^r_{p,\alpha}(\overset{+}{E^n})} \right). \tag{8}$$

It is not difficult to show that B is no greater than the right-hand side of inequality (8); therefore

$$\|f\|_{L_{\bar{p}}(p,p_1,\ldots,p_m)(E^m)} \ll c\left(h^{-(\delta+\rho_s\sigma_s)}\|f\|_{L_p(\overset{+}{E}{}^n)} + h^{\varepsilon-\rho_s\sigma_s}\|f\|_{L^r_{p,\alpha}(\overset{+}{E}{}^n)}\right).$$

Hence, putting $h = 1$, we obtain the required result.

Corollary. If $\varepsilon - \rho_s\sigma_s = 0$, then as $h \to \infty$

$$\|f\|_{L_{\bar{p}}(p,p_1,\ldots,p_m)(E^m)} \ll c\|f\|_{L^r_{p,\alpha}(\overset{+}{E}{}^n)}.$$

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.