ON THE THEORY OF DIMENSION

MATHEMATICS

1968

SovietRxiv

Abstract

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UDC 513.831

MATHEMATICS

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(Presented by Academician A. N. Tikhonov on 11 V 1967)

In dimension theory an essential place is occupied by the question of the coincidence of the principal dimensional invariants of a given topological space X. These invariants are **: on the one hand, the dimension defined by means of coverings ($\dim X$), and on the other hand the small ($\operatorname{ind} X$) and large ($\operatorname{Ind} X$) dimensions, defined inductively (in the first case the induction is over points, and in the second over closed sets).

In the class of metric spaces of countable weight (in particular, if X is compact), all three invariants named above coincide ([8,7,2]).

However, this harmony is sharply disturbed as soon as we turn to spaces of a more general nature. In particular, an example is known of a metric space X such that $\operatorname{ind} X = 0$, whereas $\dim X = 1$ ([6]). True, the equality $\operatorname{Ind} X = \dim X$ in the class of metric spaces nevertheless holds ([3]). The situation is worse with the class of bicompacta: the dimension defined by means of coverings may turn out to be less than the small (and, a fortiori, the large) inductive dimension (see, for example, ([5]), or ([4]))***. Thus, in the general case of normal spaces one cannot assert the coincidence of any two of the three invariants named above.

Meanwhile, the works of P. S. Aleksandrov (for example, ([1])) convincingly show that it is precisely the class of normal spaces that is the natural object for a meaningful construction of dimension theory. In this, the definition of dimension by means of coverings is taken as the foundation of the theory.

What has been said, it seems to me, justifies an attempt to redefine the invariants $\operatorname{ind} X$ and $\operatorname{Ind} X$ in such a way that their coincidence with $\dim X$ would hold for as broad a class of spaces as possible****. It is precisely such an attempt that is undertaken in the present note. The essence of the proposed redefinition consists approximately in the following: restrictions of dimensional type are imposed not on the separating sets themselves (as is always done in the inductive definition of dimension), but on sets approximating them. As the approximating apparatus, ω-maps are used. In this way two inductively defined dimensional invariants are obtained, which will be denoted by $\operatorname{ind}^* X$ and $\operatorname{Ind}^* X$. The first of them corresponds to the small inductive dimension, and the second to the large.

The equality $\operatorname{Ind}^* X = \dim X$ turns out to be valid in the class of normal spaces, and the equality $\operatorname{ind}^* X = \dim X$ in the class of bicompacta.

Before passing to precise formulations of the definitions and results, let us agree on the following:

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* Reported in 1966 at the International Congress of Mathematicians in Moscow and at the Second Prague Topological Symposium.

** Connections with homological dimensions are not discussed.

*** The question of the coincidence, for an arbitrary bicompactum X, of the invariants $\operatorname{ind} X$ and $\operatorname{Ind} X$ remains open. It is unlikely that it is settled positively.

**** I first heard such a formulation of the question from Yu. M. Smirnov.

1. The term "space" means a normal topological space.
2. By a covering of a space we mean a finite open covering.
3. The boundary of an open set V in a space X will be denoted by $\dot{V}$: $\dot{V} = [V] \setminus V$.
4. Let F be a closed subset of a space X lying in an open set U. An open set V satisfying the inclusions $F \subseteq V \subseteq U$ will sometimes be called, for brevity, an (F, U)-neighborhood. (In particular, an (F, X)-neighborhood is any open set in X containing F.)

The definition of the invariants $\operatorname{Ind}^* X$ and $\operatorname{ind}^* X$ is based (as has already been said) on the well-known notion of an ω-mapping. We shall now define this notion in a form convenient for what follows.

Let X, Y be spaces, let X_0 be a normal subspace of the space X, and let $\omega = \{O_1, O_2, \dots, O_s\}$ be a system of open subsets of X forming a covering of the space X_0. A continuous mapping f of the space X_0 onto Y is called an ω-mapping if for every point $y_0 \in Y$ there exist a neighborhood U of it and an element O_k of the system ω such that $f^{-1}U \subseteq O_k$.

We now pass to precise formulations of the definitions and results.

Preliminary definition. Let $\mathfrak{X}$ be a class of spaces, let X be a space, and let ω be a covering of this space. By definition, X is ω-close to $\mathfrak{X}$ if there exist $Y \in \mathfrak{X}$ and an ω-mapping of the space X onto Y.

Definition of the invariant $\operatorname{Ind}^* X$.

1. $\operatorname{Ind}^* X = -1$ if and only if X is empty.
2. Suppose that, for an arbitrary nonnegative integer n, the class $\mathfrak{X}_{n-1}$ of those spaces X for which $\operatorname{Ind}^* X \le n-1$ has already been defined.

3. $\operatorname{Ind}^* X \leq n$ if, for every closed $F \subseteq X$, every (F, X)-neighborhood U, and every covering ω of the space X, there exists an (F, U)-neighborhood V whose boundary $\dot{V}$ is ω-close to $\mathfrak{X}_{n-1}$.
4. The class of those spaces X for which $\operatorname{Ind}^* X \leq n$ is denoted by $\mathfrak{X}_n$.
5. $\operatorname{Ind}^* X = n$ if $X \in \mathfrak{X}_n \setminus \mathfrak{X}_{n-1}$.
6. $\operatorname{Ind}^* X = \infty$ if the inclusion $X \in \mathfrak{X}_n$ is not fulfilled for any integer $n \geq -1$.

Theorem 1. *For every normal space X the equality* $\operatorname{Ind}^* X = \dim X$ *holds.*

Let us note that if in item 3 of the definition given above, instead of requiring ω-closeness of the set $\dot{V}$ to the class $\mathfrak{X}_{n-1}$, one required the inclusion $V \in \mathfrak{X}_{n-1}$, then the usual definition of the large inductive dimension would be obtained.

In defining the invariant $\operatorname{ind}^* X$, a certain caution is needed: the automatic replacement of the closed set $F \subseteq X$ appearing in the definition of the invariant $\operatorname{Ind}^* X$ by a point $x \in X$ would lead to the inequality $\operatorname{ind}^* X \leq 1$ for every space X. Indeed, let U be a neighborhood of the point x, and let ω be a covering of the space X. Choose a neighborhood V of the point x so that its closure is contained not only in U, but also in one of the elements of the covering ω. It is clear that the boundary of this neighborhood is ω-close to the class consisting of two spaces: the one-point space and the empty space.

From what has been said it is evident that, in defining the invariant $\operatorname{ind}^* X$, one must restrict V not only "from above" (by the set U), but also "from below" (by some open set W depending on U, but not depending on ω).

Now let us formulate.

Definition of the invariant $\operatorname{ind}^* X$.

1. $\operatorname{ind}^* X = -1$ if and only if X is empty.
2. Suppose that, for an integer nonnegative n, the class $\tilde{\mathfrak{x}}_{n-1}$ of those spaces X for which $\operatorname{ind}^* X \leq n-1$ has already been defined.
3. $\operatorname{ind}^* X \leq n$ if, for every point $x \in X$ and every neighborhood U of it, there is an (x, U)-neighborhood W such that, for an arbitrary cover ω of the space X, there exists a $([W], U)$-neighborhood V whose boundary $\dot{V}$ is ω-close to the class $\tilde{\mathfrak{x}}_{n-1}$.
4. The class of those spaces X for which $\operatorname{ind}^* X \leq n$ is denoted by $\tilde{\mathfrak{x}}_n$.
5. $\operatorname{ind}^* X = n$ if $X \in \tilde{\mathfrak{x}}_n \setminus \tilde{\mathfrak{x}}_{n-1}$.
6. $\operatorname{ind}^* X = \infty$ if the inclusion $X \in \tilde{\mathfrak{x}}_n$ is not satisfied for any integer $n \geq -1$.

Theorem 2. *For every bicompactum X the equality*

$$\operatorname{ind}^* X = \dim X$$

holds.

If, in item 3 of the definition of the invariant $\operatorname{ind}^* X$, instead of requiring the ω-closeness of the set $\dot{V}$ to the class $\tilde{\mathfrak{x}}_{n-1}$ one requires the inclusion $\dot{V} \in \tilde{\mathfrak{x}}_{n-1}$, then the usual definition of small inductive dimension is obtained. The mention of the set W then acquires a parasitic character; as is easy to see, it does not affect the meaning of the definition.

Received
11 V 1967

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Note: Figure translations are in progress. See original paper for figures.