PS-ISOMORPHISM OF A STRUCTURALLY ORDERED GROUP

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PS-ISOMORPHISM OF A STRUCTURALLY ORDERED GROUP

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In this paper it is proved that a structurally ordered group is determined by the structure of its subsemigroups.

We shall use the following definitions, introduced in the papers ([1,2]). Let a one-to-one correspondence φ be established between the elements of two arbitrary groups G and G^{φ}, and let the element $g \in G$ correspond to the element $\varphi(g) \in G^{\varphi}$. We shall say that an element a is directly (respectively, inversely) φ-parallel to an element b, or that the elements a, b are directly (respectively, inversely) φ-parallel, if $\varphi(ab) = \varphi(a)\varphi(b)$ (respectively, if $\varphi(ab) = \varphi(b)\varphi(a)$). If for the element a at least one of the definitions (direct or inverse) of φ-parallelism to the element b is satisfied, then the element a is called φ-parallel to the element b, or the elements a, b are called φ-parallel. Finally, if the element a is (directly, inversely) φ-parallel to the element b, and the element b is (directly, inversely) φ-parallel to the element a, then the elements a and b will be called mutually (directly, inversely) φ-parallel.

If any two elements of the group G are directly φ-parallel, then the groups G and G^{φ} are isomorphic, and the correspondence φ is an isomorphism of the groups G and G^{φ}. If any two elements of the group G are inversely φ-parallel, then the groups G and G^{φ} are anti-isomorphic, and the correspondence φ is an anti-isomorphism of the groups G and G^{φ}. Finally, if any two elements of the group G are φ-parallel, then, following the paper ([3]), we shall say that the groups G and G^{φ} are semi-isomorphic, and the correspondence φ will be called a semi-isomorphism.

The paper considers an isomorphism φ of the structure $P(G)$ of all subsemigroups of a structurally ordered group G onto the structure $P(G^{\varphi})$ of all subsemigroups of the group G^{φ} (a PS-isomorphism of the groups G and G^{φ}). The empty set, as usual, is regarded as a subsemigroup of every semigroup. A PS-isomorphism of the groups G and G^{φ} is denoted by the letter φ. If A is a subsemigroup of G, then A^{φ} denotes its image under the PS-isomorphism φ. Since a structurally ordered group is a torsion-free group $((^4), theorem 7, p.304)$, it is known that the image $\{g\}^{\varphi}$ of the cyclic subsemigroup $\{g\}$ under the PS-isomorphism φ is a cyclic subsemigroup (see ([5]), theorem 2.12). Therefore in what follows we shall assume that the PS-isomorphism φ establishes between the elements of the groups G and G^φ a correspondence φ under which to an element $g \in G$ there corresponds an element $\varphi(g) \in G^\varphi$ such that $\{\varphi(g)\} = \{g\}^\varphi$. This correspondence, as is easy to see, is one-to-one. By $\varphi(M)$ we shall denote the set of images of all elements of the set $M \subset G$ under the mapping φ. It is also known that two permutable elements a and b of the group G are φ-parallel and $\varphi(g^n) = \varphi(g)^n$. From the permutability of the elements a and b in the group G there follows the permutability of their images $\varphi(a)$ and $\varphi(b)$ in the group G^φ, and conversely (see ([6]), theorem 2.9).

Let φ be a PS-isomorphism of the groups G and G^φ, and let a one-to-one correspondence φ be established between the elements of the groups G and G^φ. We shall say that the PS-isomorphism φ is induced by the mapping φ of the group G onto the group G^φ, if for every subsemigroup P of the group G, $P^\varphi = \varphi(P)$ (see ([6]), p. 71).

Let the structures of elements of the structurally ordered groups G and G^φ be isomorphic, and let between the elements of the groups G and G^φ there be established a mutu-

a one-to-one correspondence φ. We shall say that an isomorphism of the lattices of elements of the lattice-ordered groups G and G^φ is a consequence of the mapping φ, if for any two elements $a, b \in G$ the following hold:

$$\varphi(a \vee b) = \varphi(a) \vee \varphi(b), \qquad \varphi(a \wedge b) = \varphi(a) \wedge \varphi(b).$$

Let Σ be a lattice. Let X be an element of Σ. We shall call the element X singly covered ([7]) if the set of all elements of the lattice Σ that are less than or equal to X has a greatest element distinct from X. A singly covered element in the lattice $P(G)$ of a torsion-free group G is a cyclic subsemigroup (([5]), Lemma 2.11), and conversely, every cyclic subsemigroup is a singly covered element (([5]), Lemma 2.10). In addition, we shall need the following notions.

We shall call an element $f \in \{x, y\}$ the C-product of the elements x and y, if the subsemigroup $\{x, y\}$ contains more than two elements and if, whenever $u, v \in \{x, y\}$ and $u \neq f \neq v$, from $f \in \{u, v\}$ it follows that $u = x,\ v = y$ or $u = y,\ v = x$.

Let Σ be a lattice, and let X, Y, U, V, F be singly covered elements of Σ. We shall call an element $F < X \vee Y$ a C-element for the elements X and Y, if, whenever $U < X \vee Y$ and $V < X \vee Y$, and $U \neq F \neq V$, from $F < U \vee V$ it follows that $U = X,\ V = Y$ or $U = Y,\ V = X$. (For analogous definitions, see ([8]).)

If the product xy of elements x and y of a semigroup Γ is not representable in the form of any word in the alphabet x, y different from xy and yx, then we shall call the product xy unique ([2]).

Lemma 1. *The product xy of any two positive elements $x \neq e$ and $y \neq e$ of a partially ordered group G is unique.*

Lemma 2. *If the product xy of elements x and y of a semigroup Γ is unique, then the element xy is a C-product of the elements x and y.*

Proof. Suppose that the element xy is not a C-product; then there exist words u and v in the alphabet x, y, of length one and equal to each other or of length greater than one, such that $u \neq xy \neq v$ and $xy \in \{u, v\}$. In the first case, if u and v are words of length one and are equal, the element xy obviously has a representation different from xy and yx. Consider the second case, when u and v are words of length greater than one in the alphabet x, y. By assumption, the word xy in the alphabet u, v has length greater than or equal to two (i.e., neither of the equalities $xy = u = yx$ and $xy = v = yx$ is possible). Therefore the element xy has a representation different from xy and yx.

Lemma 3. *Let x, y be elements of a torsion-free group G. An element $f \in G$ is then and only then a C-product of the elements x and y, when the element $F = \{f\}$ of the lattice $P(G)$ is a C-element for the elements $X = \{x\}$ and $Y = \{y\}$.*

Lemma 4. *Let φ be a PS-isomorphism of a torsion-free group G and the group G^φ. If an element $f \in G$ is a C-product of the elements x and y, then $\varphi(f)$ is a C-product of the elements $\varphi(x)$ and $\varphi(y)$, and conversely.*

Lemma 5. *If an element f is a C-product of the elements x and y of the group G, then $f = xy$ or $f = yx$.*

Lemmas 3, 4, and 5 are analogous to the corresponding assertions from ([8]). The proofs of these lemmas are not difficult to carry out directly.

Lemma 6. *Let a torsion-free partially ordered group G be PS-isomorphic to the group G^φ. Then the mapping φ, established between the elements of the groups G and G^φ by the PS-isomorphism Φ, is an isomorphism or an anti-isomorphism of the semigroups P and P^φ, where P is the subsemigroup of positive elements of the group G.*

Proof. First we shall show that arbitrary positive elements a and b of the group G are mutually φ-parallel. By Lemma 1, the product ab is unique. By Lemma 2, the product ab is a C-product of the elements a and b. By Lemma 4, the element $\varphi(ab)$ is a C-product of the elements $\varphi(a)$ and $\varphi(b)$, and hence, by Lemma 5,

$$\varphi(ab) = \varphi(a)\varphi(b)$$

or

$$\varphi(ab) = \varphi(b)\varphi(a).$$

In the same way we obtain

$$\varphi(ba) = \varphi(b)\varphi(a)$$

or

$$\varphi(ba) = \varphi(a)\varphi(b).$$

Thus, the semigroups P and P^φ are semi-isomorphic. Further, by Theorem 1 of [3], we obtain that the semigroups P and P^φ are isomorphic or anti-isomorphic.

Lemma 7. *Let G be a partially ordered group with semigroup of positive elements P. The group G will be structurally ordered if and only if, for any two elements $a, b \in G$, there is an element $c \in G$ such that the equality $Pa \cap Pb = Pc$ holds.*

Proof. Let G be a structurally ordered group with semigroup of positive elements P, and let a, b be arbitrary elements of G. Let $a \vee b = c$; then from $x \in Pc$ it follows that $x > c$, and therefore $x > a$ and $x > b$, i.e. $Pc \subset Pa \cap Pb$. Conversely, if $x \in Pa \cap Pb$, then $x > a$ and $x > b$, i.e. $x > c$ and $x \in Pc$. Thus, $Pa \cap Pb = Pc$.

Suppose that for any two elements a and b of G there is an element c such that $Pa \cap Pb = Pc$. In particular, for $b = e$, where e is the identity of the group, $Pa \cap Pe = Pc$. Hence the element c will be the least upper bound of the elements a and e, i.e. $c = a \vee e$, and, by Theorem 2 ([9], p. 298), the group G is structurally ordered.

Lemma 8. *Let G be a structurally ordered group with semigroup of positive elements P; then the equalities $a \vee b = c$ and $Pa \cap Pb = Pc$, and also the equalities $a \wedge b = c$ and $Pa^{-1} \cap Pb^{-1} = Pc^{-1}$, are equivalent.*

Proof. We prove the first assertion; the second is proved analogously. Let $a \vee b = c$; then from $x \in Pc$ it follows that $x > c$, and therefore $x > a$ and $x > b$, i.e. $Pc \subset Pa \cap Pb$. Conversely, if $x \in Pa \cap Pb$, then $x > a$ and $x > b$, i.e. $x > c$ and $x \in Pc$. Thus, $Pc = Pa \cap Pb$. Now suppose $Pa \cap Pb = Pc$. It follows that $c > a$ and $c > b$, i.e. $c > a \vee b$, $a \vee b > b$, $a \vee b > a$; consequently, $a \vee b \in Pa \cap Pb = Pc$, i.e. $a \vee b > c$. Thus, $a \vee b = c$.

Theorem. *Let φ be a PS-isomorphism of a structurally ordered group G onto a group G^φ. Then the groups G and G^φ are isomorphic both as groups and as structures of elements, and the isomorphism of structures of elements and the PS-isomorphism is a consequence of an isomorphism or anti-isomorphism of groups.*

Proof. A structurally ordered group is a group without torsion (([4], theorem, 7, p. 304); therefore, by Lemma 6, the mapping φ, established between the elements of the groups G and G^φ by the PS-isomorphism φ, is an isomorphism or anti-isomorphism of the semigroups P and P^φ, where P is the semigroup of positive elements of the group G.

Suppose that φ is an isomorphism of the semigroups P and P^φ. In the case where φ is an anti-isomorphism, the proof is analogous.

1. First let us prove that every positive element a is directly φ-parallel to any element $b \in G$. Since the group G is structurally ordered, any element g of it is written in the form of a product

$$g = g_1 g_2^{-1},$$

where g_1 and g_2 are positive permutable elements (see, for example, [4], p. 303, § 4). Let $ab = c$ and $b = b_1 b_2^{-1}$, $c = c_1 c_2^{-1}$, where $b_1, b_2, c_1, c_2 \in P$ and $b_1 b_2 = b_2 b_1$, $c_1 c_2 = c_2 c_1$. Then

$$a b_1 b_2^{-1} = c_1 c_2^{-1}, \qquad c_2 a b_1 = c_1 b_2, \qquad \varphi(c_2 a b_1) = \varphi(c_1 b_2).$$

By Lemma 6,

$$\varphi(c_2)\varphi(a)\varphi(b_1) = \varphi(c_1)\varphi(b_2)$$

and

$$\varphi(a)\varphi(b_1)\varphi(b_2)^{-1} = \varphi(c_2)^{-1}\varphi(c_1).$$

Further, from the permutability of b_1 and b_2^{-1}, and of c_1 and c_2^{-1}, it follows that

$$\varphi(b) = \varphi(b_1 b_2^{-1}) = \varphi(b_1)\varphi(b_2^{-1}) = \varphi(b_1)\varphi(b_2)^{-1},$$

and also

$$\varphi(c) = \varphi(c_1)\varphi(c_2)^{-1}.$$

From the permutability of c_1 and c_2^{-1} follows the permutability of their images $\varphi(c_1)$ and $\varphi(c_2^{-1}) = \varphi(c_2)^{-1}$. Consequently,

$$\varphi(c) = \varphi(c_2)^{-1}\varphi(c_1).$$

Thus, we obtain

$$\varphi(a)\varphi(b) = \varphi(c) = \varphi(ab).$$

2. Now let us show that any two elements $a, b \in G$ are directly φ-parallel. Let $ab = c$. Since the group G is structurally ordered, we may put

$$a = a_1 a_2^{-1}, \qquad b = b_1 b_2^{-1},$$

where $a_1, a_2, b_1, b_2 \in P$ and

$$a_1 a_2 = a_2 a_1, \qquad b_1 b_2 = b_2 b_1,$$

whence

$$a_1 a_2^{-1} b_1 b_2^{-1} = c$$

and

$$a_1 b_1 = a_2 c b_2.$$

$$\varphi(a_1 b_1) = \varphi(a_2 c b_2).$$

By item 1,

$$\varphi(a_1)\varphi(b_1) = \varphi(a_2)\varphi(c b_2) = \varphi(a_2)\varphi(c)\varphi(b_2),$$

$$\varphi(a_2)^{-1}\varphi(a_1)\varphi(b_1)\varphi(b_2)^{-1} = \varphi(c).$$

But, just as in item 1, we have

$$\varphi(a) = \varphi(a_1 a_2^{-1}) = \varphi(a_1)\varphi(a_2^{-1}) = \varphi(a_1)\varphi(a_2)^{-1} = \varphi(a_2)^{-1}\varphi(a)$$

and

$$\varphi(b) =$$

$= \varphi(b_1)\varphi(b_2)^{-1}$. Therefore $\varphi(a)\varphi(b) = \varphi(c) = \varphi(ab)$, as was required to prove.

Thus, the groups G and G^φ are isomorphic, and the PS-isomorphism φ is a consequence of an isomorphism or anti-isomorphism φ of the groups G and G^φ.

3. Let us show that the groups G and G^φ are isomorphic as structures of elements, and that this isomorphism is a consequence of the mapping φ, i.e. of an isomorphism or anti-isomorphism of groups. For this it is enough to show that if the semigroup P^φ is taken as the semigroup of positive elements, then the group G^φ will be structurally ordered, and for any two elements $a, b \in G$ the two equalities will hold

$$1) \quad \varphi(a \vee b) = \varphi(a) \vee \varphi(b), \qquad 2) \quad \varphi(a \wedge b) = \varphi(a) \wedge \varphi(b).$$

We shall prove the first equality; the second is proved analogously. We carry out the proof for the case of an isomorphism of the groups G and G^φ. In the case of an anti-isomorphism the proof is analogous. By Lemma 8, the equality $a \vee b = c$ is equivalent to the equality $Pa \cap Pb = Pc$. Let us show that $\varphi(Pc) = P^\varphi\varphi(c)$. Indeed, if $g \in Pc$, then the element g has the form $g = pc$, where $p \in P$. Further, in view of the isomorphism of the groups G and G^φ, we have $\varphi(g) = \varphi(pc) = \varphi(p)\varphi(c) \in P^\varphi\varphi(c)$, i.e. $\varphi(Pc) \subset P^\varphi\varphi(c)$; in the same way we show that $P^\varphi\varphi(c) \subset \varphi(Pc)$. Thus, $\varphi(Pc) = P^\varphi\varphi(c)$. Now let us show that $\varphi(Pa \cap Pb) = P^\varphi\varphi(a) \cap P^\varphi\varphi(b)$. Indeed, if $g \in Pa \cap Pb$, this means that the element g is representable in the form $g = p_1 a = p_2 b$, where $p_1, p_2 \in P$. Further,

$$\varphi(g) = \varphi(p_1 a) = \varphi(p_2 b) = \varphi(p_1)\varphi(a) = \varphi(p_2)\varphi(b) \in P^\varphi\varphi(a) \cap P^\varphi\varphi(b).$$

Consequently, $\varphi(Pa \cap Pb) \subset P^\varphi\varphi(a) \cap P^\varphi\varphi(b)$. We similarly show that $P^\varphi\varphi(a) \cap P^\varphi\varphi(b) \subset \varphi(Pa \cap Pb)$. Thus, $\varphi(Pa \cap Pb) = P^\varphi\varphi(a) \cap P^\varphi\varphi(b)$. Consequently, the equality $Pa \cap Pb = Pc$ entails the equality

$$P^\varphi\varphi(a) \cap P^\varphi\varphi(b) = P^\varphi\varphi(c).$$

Since the groups G and G^φ are isomorphic, the subsemigroup P^φ will be invariant in the group G^φ and may be taken as the semigroup of positive elements. By Lemma 7, the group G^φ in this case turns out to be structurally ordered and, by Lemma 8, $\varphi(a) \vee \varphi(b) = \varphi(c) = \varphi(a \vee b)$. Thus, the theorem is completely proved.

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REFERENCES

1. K. M. Kutyev, DAN, 135, No. 6, 1326 (1960).
2. K. M. Kutyev, Izv. AN SSSR, ser. matem., 27, 701 (1963).
3. W. R. Scott, Proc. Am. Math. Soc., 8, No. 6, 1141 (1957).
4. G. Birkhoff, *Lattice Theory*, IL, 1952.
5. R. V. Petropavlovskaya, Matem. sborn., 28 (70), 3, 589 (1951).
6. R. V. Petropavlovskaya, Matem. sborn., 29 (71), 1, 63 (1951).
7. L. N. Shevrin, Sibirsk. matem. zhurn., 3, No. 3, 446 (1962).
8. L. N. Shevrin, Sibirsk. matem. zhurn., 5, No. 3, 671 (1964).

Note: Figure translations are in progress. See original paper for figures.