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Abstract

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CYBERNETICS AND CONTROL THEORY

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ON TOPOLOGICAL PRINCIPLES OF SELF-CORRECTION

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In (1) the problem of synthesizing self-correcting circuits was posed, and it was shown that correction of a single short circuit in contact circuits does not lead to an increase in the asymptotics of the Shannon function. An analogous result for contact circuits with open circuits was obtained in (2). However, when the number of errors is large, the methods of these works do not apply. In (3) it was shown that the problem of self-correction in circuits made of functional elements can be solved with the aid of self-correcting codes. However, codes are not applicable to gate and contact circuits, since substitutions are not allowed in such circuits. Therefore, for gate and contact circuits the problem posed required the search for other means.

In the present note, gate and contact circuits with open circuits and short circuits of edges are considered. Two principles of self-correction, not connected with the use of codes, are presented. With their help it is possible to correct an increasing number of errors without increasing the asymptotics of the Shannon function, or increasing it by no more than a factor of two. The synthesis methods found can easily be modeled by circuits of other types.

1°. We shall say that a gate circuit (4) **corrects an open circuit of a edges** if the matrix it realizes is not changed when any no more than a edges in the circuit are opened. The **depth** of a circuit is the maximum number of gates forming an oriented chain. Let $L(\mathfrak{A})$ denote the number of gates in the circuit $\mathfrak{A}$; let $L_a^{(m)}(F)$ denote the minimum number of gates sufficient to realize the matrix F by a gate circuit of depth not exceeding m, correcting an open circuit of a edges; let $L_a^{(m)}(\mathfrak{F})$ denote $\max L_a^{(m)}(F)$, where the maximum is taken over all matrices from the class of matrices $\mathfrak{F}$. The **redundancy coefficient** is the number

$$ {}_{a}^{(m)}(\mathfrak{F}) = L_a^{(m)}(\mathfrak{F})/L_0^{(m)}(\mathfrak{F}). $$

Analogously, for circuits of arbitrary depth the number ${}_{a}(\mathfrak{F})$ is defined.

A matrix having p rows and q columns will be called a (p, q)-**matrix**. The **density** of a matrix is the fraction of ones in the total number of its elements. Let $\mathfrak{F}(p, q, \alpha)$ denote the class of all (p, q)-matrices of density α.

Theorem 1 (the 1st principle of self-correction). *Let the conditions be satisfied:*

a) $q_n \le p_n$;

b) $q_n \to \infty$;

c) $(1 - \alpha_n)q_n / \log p_n \to \infty$;

d) $\log p_n \Big/ \log \dfrac{1}{1-\alpha_n} \to \infty$;

d) $a_n \to 1$;

e)

$$ a_n \Big/ \min \left\{ \frac{\log \dfrac{1}{1-a_n}}{\log\log \dfrac{1}{1-a_n}}, \frac{\log p_n}{\log \dfrac{1}{1-a_n}} \right\} \to 0. $$

Then*

$$ I_{a_n}^{(2)}(\mathfrak{F}(p_n, q_n, a_n)) \sim 1. $$

Theorem 2. Suppose that the condition

a) $\log p_n \sim \log q_n$

and conditions b), c), d), e) of Theorem 1 are satisfied. Then, on gate circuits of depth 3, the estimate

$$ I_{a_n}(\mathfrak{F}(p_n, q_n, a_n)) \sim 1 $$

is attained.

2°. We say that a contact or contact-gate circuit **corrects breaks of edges and closures of edges** if the realized function is not changed when arbitrary no more than a edges are broken in the circuit and, simultaneously, other arbitrary no more than b edges are closed. Let $L_{a,b}(n)$ denote the minimum number of contacts sufficient for realizing an arbitrary Boolean function of n arguments by a contact circuit that corrects the breaking of a edges and the closure of b edges.

The **redundancy coefficient** is the number

$$ I_{a,b}(n) = L_{a,b}(n)/L_{0,0}(n). $$

Theorem 3.** For contact circuits, when $a_n = O(\log n / \log\log n)$, the estimate

$$I_{a_n,0}(n) \sim 1$$

holds.

The **proof** is based on representing every function in the form of a conjunction of "dense" functions, to which the first principle is applied.

Theorem 4. For contact Π-circuits:

a) if $a_n = o((\log n / \log\log n)^{1/2})$, then $I_{a_n,0}(n) \sim 1$;

b) if $b_n = o((\log n / \log\log n)^{1/2})$, then $I_{0,b_n}(n) \sim 1$.

The **proof** of item a) is obtained by the reduction indicated above to the first principle; item b) is reduced to item a) by a simple application of the principle of duality.

3°. We describe the second principle in its simplest form.

We shall consider a contact circuit of the form shown in Fig. 1. The input pole of the circuit is the extreme left node, and the output poles are all nodes lying on the line O_4O_4'. To the left of the line O_2O_2' the circuit is constructed in the same way as the circuit of O. B. Lupanov ([6],[7]) for conjunctions

$$K_\sigma(\tilde{x}) = x_1^{\sigma_1} x_2^{\sigma_2} \dots x_k^{\sigma_k}.$$

Between the lines O_1O_1', O_2O_2' there are "brooms" of contacts, cutting spheres. Between the lines O_2O_2', O_3O_3' short connections are inserted. They join only those nodes on the line O_2O_2' which correspond to one sphere. We denote by $\mathfrak{B}$ the subcircuit enclosed between the lines O_1O_1', O_3O_3'. The **distance** between nodes of the subcircuit $\mathfrak{B}$ is the minimum number of "brooms" through which a chain passes that connects these nodes and is located entirely in the subcircuit $\mathfrak{B}$. We say that nodes lying on the line O_3O_3' are ***b*-colored** in the subcircuit $\mathfrak{B}$ if any two such nodes whose distance in the subcircuit $\mathfrak{B}$ is not greater than b are colored in different colors. The cir-

* If $a_n < 1 - \varepsilon$, where ε is a positive constant, then $I^{(2)}_{a_n} > a_n^{(5)}$.

** This theorem contains the result of Kh. A. Madatyan ([2]) as a special case when $a_n = 1$.

We call a circuit of the kind under consideration **b-colored** if: 1) every contact not contained in the subcircuit $\mathfrak{B}$ is duplicated successively $b+1$ times; 2) the nodes located on the line O_3O_3' are b-colored in the subcircuit $\mathfrak{B}$; 3) any two d.n.f.' s over $\tilde{y}$ are orthogonal.

We extend continuously the coloring from the nodes located on the line O_3O_3' to the lines O_2O_2', O_4O_4', without crossing the latter. Color each conjunction $K_\sigma(\tilde{x})$ issued by a node on the line O_2O_2' in the color of this node.

Fig. 1

Figure 1: Fig. 1

Fig. 1

By $\Phi_w(\tilde{x})$ we denote the disjunction of all conjunctions colored in the color w. It is easy to see that any two functions $\Phi_w(\tilde{x})$ are orthogonal. A **corrector** of the b-colored circuit W is a contact $(W, 1)$-circuit realizing the system of functions $\{\Phi_w\}$ and correcting the closure of b contacts. Correction of closures in the corrector is achieved by duplicating each contact $b+1$ times. By a **connection of a b-colored circuit with its corrector** we mean the two-terminal circuit obtained as a result of adjoining all output poles of the colored circuit that are colored in the color w to the input pole of the corrector on which the function Φ_w is realized; the input pole of the connection is the input pole of the colored circuit, and the output pole is the output pole of the corrector.

Theorem 5 (2nd principle of self-correction). *The connection of a b-colored circuit with its corrector realizes the disjunction of all functions issued by the output poles of the colored circuit, and corrects the closure of b contacts.*

4°. With the aid of the 2nd principle the following are proved.

Theorem 6*. *For contact circuits, when*

$$b_n = O((n/\log n)^{1/2}),$$

the estimate

$$I_{0,b_n}(n) \lesssim 2$$

is valid.

Theorem 7. *For contact circuits, when*

$$a_n = o(\log n/\log\log n),$$

$$b_n \leq n^{1/2-\varepsilon},$$

where ε is an arbitrarily small positive constant, the estimate

$$I_{a_n,b_n}(n) \lesssim 2$$

is valid.

Theorem 8. *For contact circuits, when*

$$a_n = o(\log n/\log\log n),$$

the estimate

$$I_{a_n,3}(n) \sim 1$$

is valid.

In proving Theorems 7, 8, both principles are applied simultaneously. In proving Theorem 8, item 3) in the definition of a colored circuit is replaced by the device from (1).

By the **redundancy coefficient** $I_{a,b}(n)$ **for almost all functions** we mean the number $I_{a,b}(\mathfrak{F})$ for a class $\mathfrak{F}$ of Boolean functions of n arguments such that

$$\varkappa(\mathfrak{F})/2^{2^n} \to 1,$$

where $\varkappa(\mathfrak{F})$ is the number of functions in the class $\mathfrak{F}$.

* This theorem strengthens a result of Yu. G. Potapov and S. V. Yablonskii (1).

Theorem 9. For contact circuits and for almost all functions, if $a_n = o((n/\log n)^{1/2})$, the estimate

$$I_{a_n,0}(n) \sim 1.$$

is valid.

5°. By L_{cont} we denote the weight of a contact, and by L_{vent} the weight of a gate in a contact-gate circuit; we assume that these numbers are positive. The complexity of a circuit is defined as the sum of the weights of all its edges.

Theorem 10. For contact-gate circuits, if $L_{\text{vent}} \leq L_{\text{cont}}$,
$a_n = o((n/\log n)^{1/2})$,
$b_n = o((n/(a_n \log a_n + \log n))^{1/2})$, the estimate

$$I_{a_n,b_n}(n) \sim 1.$$

is valid.

Theorem 11. For contact-gate circuits, if $L_{\text{vent}} > L_{\text{cont}}$,
$a_n = o(\log n/\log\log n)$,
$b_n = o((n/\log n)^{1/2})$, the estimate

$$I_{a_n,b_n}(n) \sim 1.$$

is valid.

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Note: Figure translations are in progress. See original paper for figures.