ON THE CARDINALITY OF SYSTEMS OF OPEN SUBSETS IN A DYADIC BICOMPACTUM

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Abstract

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V. V. PROIZVOLOV

ON THE CARDINALITY OF SYSTEMS OF OPEN SUBSETS IN A DYADIC BICOMPACTUM

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In 1941, Shpilrain proved a theorem on the countability of an arbitrary point-countable system of open sets in a dyadic bicompactum $(^1)^*$. Here it will be proved that a dyadic bicompactum has a stronger property of the same order. Namely, the following is true.

Theorem 1. *The cardinality of an arbitrary system of open sets of a dyadic bicompactum does not exceed the cardinal number τ, if the pointwise cardinality of this system does not exceed τ^{**} (it is assumed that $\tau \geq \aleph_0$).*

Proof. In a dyadic bicompactum X let there be given some system of open sets $W = \{W_\alpha\}$, whose pointwise cardinality does not exceed τ. The bicompactum X is a continuous image of D^m: $f : D^m \to X$. If we denote $U_\alpha = f^{-1}W_\alpha$, then the system of open sets $U = \{U_\alpha\}$ in D^m has pointwise cardinality not exceeding τ. It is enough for us to prove that the cardinality of U does not exceed τ; then the cardinality of W also does not exceed τ.

Represent
$$D^m = \prod_\lambda D_\lambda,$$
where for every index λ the factor D_λ is the simple two-point space, and take in D^m the canonical base, each element of which has a finite number of distinguished indices; denote it by $B = \{B_\alpha\}$.

In each element of the system U take one element of the base B. It may happen that for distinct U_{α_1} and U_{α_2} one and the same element $B_\beta \in B$, belonging to their intersection, is chosen; in this case we shall count the element B_β only once. The totality of all such chosen elements forms a system $V = \{V_\alpha\}$, and if $\alpha \neq \beta$, then $V_\alpha \neq V_\beta$. The system V is inscribed in the system U in such a way that the pointwise cardinality of V does not exceed τ.

The systems U and V are equipotent. It is immediately clear that the cardinality of $U \geq$ the cardinality of V. Let us verify the reverse inequality. Suppose the contrary: $\tau_1 > \tau_2$, where τ_1 is the cardinality of the system U, and τ_2 is the cardinality of the system V, with $\tau_1 > \tau$ (otherwise everything would already have been proved). Since an element of the system V is inscribed in each element of the system U, there will be such an element $V_\alpha \in V$ that is inscribed simultaneously in τ_1 elements of the system U, but this contradicts the fact that the pointwise cardinality of the system U does not exceed τ. Consequently, it remains only to prove that the cardinality of the system $V = \{V_\alpha\}$ does not exceed τ.

Suppose the contrary: the cardinality of the system V is τ_1, where $\tau_1 > \tau$. If $\tau_1 > 2^\tau$, then from V one can select a subsystem of cardinality 2^τ, so we shall assume that $\tau_1 \leq 2^\tau$.

Since each element $V_\alpha \in V$ is an element of the base B in the bicompactum

$$D^m = \prod_\lambda D_\lambda,$$

V_α has a finite number of distinguished indices λ.

The union of all distinguished indices over all α forms a set A. The cardinality of A is equal to the cardinality of V, and therefore the cardinality of $A \leq 2^\tau$.

The bicompactum

$$R = \prod_{\lambda \in A} D_\lambda$$

may be regarded naturally as embedded in D^m. In this case the system V induces in R the system $\widetilde{V} = \{\widetilde{V}_\alpha\}$, where

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* A bicompactum is called dyadic if it is a continuous image of some D^τ (the product of τ "copies" of the two-point space).

** The pointwise cardinality of a system does not exceed τ if every point of the bicompactum belongs to no more than τ elements of the system.

$\tilde{V}_\alpha = V_\alpha \cap R$. The pointwise cardinality of the system $\tilde{V}$ does not exceed τ, while the cardinality of $\tilde{V}$ coincides with the cardinality of V.

Hewitt' s theorem ([3]) asserts that the density of the space D^n, where $n \leq 2^\tau$, does not exceed τ. Therefore the bicompactum R possesses a dense subset in R whose cardinality does not exceed τ. Hence the system $\tilde{V}$, being a system of pointwise cardinality $\leq \tau$ in a space of density $\leq \tau$, has cardinality $\leq \tau$. This contradicts the assumption that the cardinality of $V > \tau$. The theorem is proved*.

Corollary. *A point-countable covering of a locally bicompact dyadic space is countable.*

A theorem more general than Theorem 7 is true.

Theorem $1'$**.** *For a bicompactum coabsolute with a dyadic one, Theorem 1 is valid.*

Proof. Thus, the bicompactum X is coabsolute with the dyadic bicompactum Y. But for a pair of coabsolute bicompacta there is always a third bicompactum Z which maps irreducibly both onto X and onto Y (for a proof of this fact see, for example, ([2])). We have irreducible mappings $f : Z \to X$ and $g : Z \to Y$.

Let the bicompactum X contain a system of open sets $U = \{U_\alpha\}$ whose pointwise cardinality does not exceed τ. It is necessary to prove that the integral cardinality of the system U does not exceed τ. In the space Z consider the system of open sets $V = \{V_\alpha\}$, where $V_\alpha = f^{-1}U_\alpha$. Obviously, the cardinality of the system V is equal to the cardinality of the system U, and the pointwise cardinality of the system V does not exceed τ. Consequently, it suffices to prove that the integral cardinality of the system V does not exceed τ.

In each element V_α of the system V choose one open set W_α such that $[W_\alpha] \subseteq V_\alpha$. The resulting system $W = \{W_\alpha\}$ has pointwise cardinality not exceeding τ, and is equipotent to the system W. Obviously, the cardinality of $V \geq$ the cardinality of W. Let us verify the reverse inequality. Suppose the contrary: $\tau_1 > \tau_2$, where τ_1 is the cardinality of the system V, and τ_2 is the cardinality of the system W, with $\tau_1 > \tau$ (otherwise everything would already have been proved). Since an element of the system W is inscribed in each element of the system V, there exists an element W_α which is inscribed in τ_1 elements of the system U, but this contradicts the fact that the pointwise cardinality of the system U does not exceed τ. Thus, V and W are equipotent. It now remains to prove that the cardinality of W does not exceed τ.

Since the mapping $g : Z \to Y$ is irreducible, the set $\langle gW_\alpha \rangle$ is nonempty for every $W_\alpha \in W$, where $\langle R \rangle$ denotes the set of all interior points of R. Put $M_\alpha = \langle gW_\alpha \rangle$; the system of open sets $M = \{M_\alpha\}$ in the space Y has the same cardinality as W. We show that the pointwise cardinality of M does not exceed τ. Suppose the contrary: there is a point $y \in Y$ which belongs to more than τ elements of the system M. But since $g^{-1}M_\alpha \subseteq [W_\alpha]$, in view of the irreducibility of g, the set $g^{-1}y$ is contained in more than τ sets of the system $\bar{W} = \{[W_\alpha]\}$. But the latter conclusion contradicts the fact that the pointwise cardinality of the system $\bar{W}$ does not exceed τ.

Thus, the pointwise cardinality of M does not exceed τ, and since Y is a dyadic bicompactum, by Theorem 1 the integral cardinality of M does not exceed τ. Consequently, the integral cardinality of W does not exceed τ, which proves everything. From the proof it is clear that Theorem 1 is also valid for bicompacta that map irreducibly onto dyadic ones.

Moscow State University
named after M. V. Lomonosov

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CITED LITERATURE

[1] E. Shpilyrain, DAN, **31**, 525 (1941).
[2] S. Iliadis, S. Fomin, UMN, **21**, no. 4 (130), 47 (1966).
[3] E. Hewitt, Bull. Am. Math. Soc., **52**, 641 (1946).

[*] As B. Efimov observed, Theorem 1 can also be proved simply by using the well-known theorem of Shanin on calibers.

Note: Figure translations are in progress. See original paper for figures.