ON A METHOD FOR SOLVING RELAXATION EQUATIONS

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Abstract

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ON A METHOD FOR SOLVING RELAXATION EQUATIONS

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This note proposes a method for solving a system of nonlinear ordinary differential equations describing relaxation in a nonequilibrium gas with internal degrees of freedom. A sequence of functions is constructed that converges to the solution of the Cauchy problem uniformly on any finite interval of variation of the argument.

1°. Nonstationary spatially homogeneous motion is described by the system of equations (see ([1]))

$$dn_i/dt = \Phi_i(n) - n_i Q_i(n), \qquad i = 1, 2, \dots, r; \tag{1}$$

$$\Phi_i(n) = \sum_{k,l,m} n_k n_l P_{kl}^{im}, \qquad Q_i = \sum_{k,l,m} n_m P_{im}^{kl}; \tag{2}$$

$$^3/_2\, kT(t) \sum_{i=1}^{r} n_i(t) + \sum_{i=1}^{r} \varepsilon_i n_i(t) = E_0; \tag{3}$$

$$n_i\big|_{t=0} = n_i(0) > 0, \qquad T\big|_{t=0} = T(0) > 0,$$

where n_i is the number density of particles of species i having internal energy ε_i; P_{kl}^{im} are given nonnegative functions of the temperature T, proportional to transition probabilities. From (1) and (2) it follows, obviously, that

$$\sum_{i=1}^{r} n_i(t) = \sum_{i=1}^{r} n_i(0). \tag{4}$$

Let us first consider the special case of the system (1)–(3), when

$$\sum_{i=1}^{r} \varepsilon_i n_i(t)$$

and T are conserved separately in time (there is no exchange of energy between the internal and translational degrees of freedom). Then the problem consists in constructing a solution of the system (1), with P_{kl}^{im} being constant quantities.

Without loss of generality, we shall assume that $\sum_{i=1}^{r} n_i(0) = 1$. We transform (1) into the equivalent system of integral equations:

$$n_i(t) = n_i(0) \exp\left\{-\int_0^t Q_i(s)\,ds\right\} + \int_0^t \Phi_i(\tau) \exp\left\{-\int_\tau^t Q_i(s)\,ds\right\} d\tau \equiv V_i(n, T). \tag{5}$$

The system (5) can be solved by the usual iteration method

$$n_i^{(n)} = V_i(n^{(n-1)}, T).$$

The sequence $n_i^{(n)}$ converges to the solution on some sufficiently small time interval. It is not difficult to see that the iterations $V_i(n^{(n-1)}, T)$ do not preserve the property of the solution (4). It is proposed to correct each iteration in such a way that the conservation law (4) is satisfied in every approximation. More precisely, the following method is proposed—

method of successive approximations:

$$n_i^{(0)} = \bar{n}_i^{(0)} = \begin{cases} n_i(0), & t = 0, \\ \bar{n}_{ip}, & t > 0, \end{cases} \tag{6}$$

$$n_i^{(n)} = V_i(\bar{n}^{(n-1)}, T), \qquad \bar{n}_i^{(n-1)} = n_i^{(n-1)} \Big/ \sum_{i=1}^{r} n_i^{(n-1)}, \tag{7}$$

where $\bar{n}_p = \{\bar{n}_{1p}, \dots, \bar{n}_{rp}\}$ is the normalized $\left(\sum_{i=1}^{r} \bar{n}_{ip} = 1\right)$ solution of the system $\Phi_i(n) - n_i Q_i(n) = 0$.

Such a construction is possible if, for any pair k, l, $\sum_{i,m} P_{kl}^{im} > 0$, since from (7) there follows the inequality

$$\rho^{(n)} = \sum_{i=1}^{r} n_i^{(n)} \geq \min\left(1, \min_{k,l} \sum_{i,m} P_{kl}^{im} \Big/ \max_{i,m} \sum_{k,l} P_{im}^{kl}\right) = \rho_{\min} > 0. \tag{8}$$

It can be proved that the sequences $n_i^{(n)}$ and $\bar{n}_i^{(n)}$ converge to the solution of system (5) uniformly on any finite time interval. From equations (7) and the inequality

$$\min\left[1, \inf_t\left(\sum_i \Phi_i(\bar{n}^{(n-1)}) \Big/ \sum_i \bar{n}_i^{(n)} Q_i(\bar{n}^{(n-1)})\right)\right] \le \rho^{(n)} \le$$

$$\le \max\left[1, \sup_t\left(\sum_i \Phi_i(\bar{n}^{(n-1)}) \Big/ \sum_i \bar{n}_i^{(n)} Q_i(\bar{n}^{(n-1)})\right)\right]$$

it follows that $\lim_{n\to\infty} n_i^{(n)}(t) = \lim_{n\to\infty} \bar{n}_i^{(n)}(t) = n_i(t)$ is a solution of the original system of equations. The following estimates of the closeness of the approximate solution to the exact one can be obtained:

$$\left|n_i(t) - n_i^{(n)}(t)\right| \le \max_i\left|n_i(0) - \bar{n}_{ip}\right| \exp\{-tQ_{\min}\} \sum_{k=n}^{\infty} \frac{(Lt)^k}{k!},$$

$$\left|n_i(t) - \bar{n}_i^{(n)}(t)\right| \le \frac{r+1}{\rho_{\min}} \max_i\left|n_i(0) - \bar{n}_{ip}\right| \exp\{-tQ_{\min}\} \sum_{k=n}^{\infty} \frac{(Lt)^k}{k!}, \tag{9}$$

where L is a constant.

For some specific transition probabilities, from system (1) one can obtain a closed equation for $\sum_{i=1}^{r} \varepsilon_i n_i(t)$, whose solution, together with (3), makes it possible to find $T(t)$ (1). Then problem (1)–(3) reduces to finding the solution of system (1), in which P_{kl}^{im} are known functions of time. In this case the method described above requires no changes except replacing $\min_{i,m} \sum P_{kl}^{im}$ and $\max_{i,m} \sum P_{kl}^{im}$ by the lower and upper bounds of the corresponding functions of time. In the general case, for solving problem (1)–(8), the following method of successive approximations is proposed:

$$n_i^{(0)} = \bar{n}_i^{(0)} = \begin{cases} n_i(0), & t = 0, \\ n_{ip}(T^{(0)}), & t > 0; \end{cases}$$

$$n_i^{(n)} = V_i(\bar{n}^{(n-1)}, T^{(n-1)}), \qquad \bar{n}_i^{(n-1)} = \frac{n_i^{(n-1)}}{\rho^{(n-1)}} \sum_{i=1}^{r} n_i(0);$$

$T^{(n)}$ is determined from the equation

$${}^3\!/_2 k T^{(n)} \sum_{i=1}^{r} n_i(0) + \sum_{i=1}^{r} \varepsilon_i \bar{n}_i^{(n)} = E_0. \tag{10}$$

The sequences $n_i^{(n)}, T^{(n)}$ converge to the solution of problem (1)–(3) uniformly on any finite interval of time, if

$$E_0 > \max \varepsilon_i \sum_{i=1}^{r} n_i(0); \qquad \min_{k,l} \sum_{i,m} P_{kl}^{im}(T) > 0, \quad 0 < T < \infty. \tag{11}$$

The solution of problem (1)–(3) is unique.

If the principle of detailed balance is satisfied, then the solution of system (1)–(3) tends, as $t \to \infty$, to the solution of the system

$$\Phi_i - n_i Q_i(n) = 0; \qquad {}^3\!/_2 kT \sum_{i=1}^{r} n_i + \sum_{i=1}^{r} \varepsilon_i n_i = E_0. \tag{12}$$

This assertion is proved by Carleman' s method ([2]), using the inequality

$$\frac{dH}{dt} = \frac{1}{4} \sum_{i,k,l,m} \left[n_k n_l \left(\frac{\mu_k \mu_l}{\mu_i \mu_m} \right)^{3/2} \exp \frac{\varepsilon_k + \varepsilon_l - \varepsilon_i - \varepsilon_m}{kT} - n_i n_m \right] P_{im}^{kl} \ln \left\{ \frac{n_i n_m}{n_k n_l} \left(\frac{\mu_i \mu_m}{\mu_k \mu_l} \right)^{3/2} \exp \frac{\varepsilon_i + \varepsilon_m - \varepsilon_k -}{kT} \right. \tag{13}$$

and the equality

$$P_{kl}^{im} = \left(\frac{\mu_k \mu_l}{\mu_i \mu_m} \right)^{3/2} \exp \left\{ \frac{\varepsilon_k + \varepsilon_l - \varepsilon_i - \varepsilon_m}{kT} \right\} P_{im}^{kl}, \tag{14}$$

where μ_i is the mass of a particle of species i;

$$H = \sum_{i=1}^{r} n_i \left\{ \ln n_i \left(\frac{\mu_i}{2\pi kT} \right)^{3/2} - {}^3\!/_2 \right\}.$$

Equality (14) is a consequence of the principle of detailed balance.

2°. The solution of the equations describing a one-dimensional stationary flow in the relaxation zone of a shock wave can be obtained by the same method. The corresponding system of equations, after introducing the new functions

$$N_i(x) = n_i(x) \Big/ \sum_{i=1}^{r} n_i(x),$$

has the form

$$N_i(x) = N_i(0)\exp\left\{-\int_0^x \overline{Q}_i(s)\,ds\right\} + \int_0^x \overline{\Phi}_i(\tau)\exp\left\{-\int_\tau^x \overline{Q}_i(s)\,ds\right\}d\tau = \overline{V}_i(N,T,\rho), \tag{15}$$

$$\overline{\Phi}_i = \sum_{k,l,m} N_k N_l \Pi_{kl}^{im}, \qquad \overline{Q}_i = \sum_{k,l,m} N_m \Pi_{im}^{kl}, \qquad \Pi_{kl}^{im} = \frac{\rho^2}{C_1\mu} P_{kl}^{im},$$

$$\rho U = C_1; \qquad \frac{k}{\mu}\rho T + \rho U^2 = C_2,$$

$$\frac{\gamma}{\gamma-1}\frac{k}{\mu}T + \frac{U^2}{2} + \frac{1}{\mu}\sum_{i=1}^{r}\varepsilon_i N_i = C_3, \tag{16}$$

where μ, γ, C_i are constants. If the functions ρ, T at infinity are prescribed, then equation (16) in some domain Ω of the values of $\sum_{i=1}^r \varepsilon_i N_i$, $C = \{C_1, C_2, C_3\}$ deter-

define single-valued positive functions: $\rho\left(\sum_{i=1}^r \varepsilon_i N_i, C\right)$, $T\left(\sum_{i=1}^r \varepsilon_i N_i, C\right)$. If from equation (15) one can obtain a closed equation for $\sum_{i=1}^r \varepsilon_i N_i$, then the problem reduces to that described above. In the general case the following method of successive approximations is proposed:

$$N_i^{(0)} = \overline{N}_i^{(0)} = \begin{cases} N_i(0), & x = 0, \\ N_{ip}(T_\infty, \rho_\infty), & x > 0; \end{cases}$$

$$N_i^{(n)} = \overline{V}_i\left(\overline{N}^{(n-1)}, T^{(n-1)}, \rho^{(n-1)}\right), \qquad \overline{N}_i^{(n-1)} = N_i^{(n-1)} \Big/ \sum_{i=1}^{r} N_i^{(n-1)},$$

$$T^{(n)} = T\left(\sum_{i=1}^{r}\varepsilon_i \overline{N}^{(n-1)}, C\right), \qquad \rho^{(n)} = \rho\left(\sum_{i=1}^{r}\varepsilon_i \overline{N}^{(n-1)}, C\right).$$

Obviously,

$$\min_i \varepsilon_i \le \sum_{i=1}^{r}\varepsilon_i \overline{N}^{(n)} \le \max_i \varepsilon_i.$$

If $\min_i \varepsilon_i, C$ and $\max_i \varepsilon_i, C$ belong to the domain Ω, and if

$$\min_{k,l}\sum_{i,m} P_{kl}^{im}(T) > 0$$

for $0 < T < \infty$, then the functions

$$\sum_{i,m}^{k,l} \Pi_{kl}^{im} \left(T^{(n)}(x), \rho^{(n)}(x)\right)$$

have a nonzero lower bound and a finite upper bound with respect to n and x. Under these conditions the sequences $N_i^{(n)}$, $\rho^{(n)}$, $T^{(n)}$ converge to the solution of the system (15), (16) uniformly on any finite interval of variation of x.

As $x \to \infty$, the functions $N_i(x)$, $\rho(x)$, $T(x)$ have the limits $N_{ip}(T_\infty, \rho_\infty)$, ρ_∞, T_∞.

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Note: Figure translations are in progress. See original paper for figures.