APPROXIMATION OF A FUNCTION OF SEVERAL VARIABLES BY A SUM OF FUNCTIONS OF A SMALLER NUMBER OF VARIABLES

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Abstract

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APPROXIMATION OF A FUNCTION OF SEVERAL VARIABLES BY A SUM OF FUNCTIONS OF A SMALLER NUMBER OF VARIABLES

(Presented by Academician M. D. Millionshchikov, 3 IV 1968)

Let, in the n-dimensional parallelepiped

$$D = \{a_i \leqslant x_i \leqslant b_i,\ i = 1, 2, \dots, n\}^*$$

there be given the function

$$\chi(x_1, x_2, \dots, x_n) = \chi_1(x_1)\chi_2(x_2) \cdots \chi_n(x_n),$$

where $\chi_i(x_i)$ does not decrease and

$$\int_{a_i}^{b_i} d\chi_i(x_i) = 1.$$

We consider functions $\varphi(x_1, x_2, \dots, x_n)$ that are χ-integrable with square in the parallelepiped D.

Denote by $\sigma_n = \{x_1, x_2, \dots, x_n\}$ the collection of n independent variables; σ_q^i (or $\sigma^i,\ \sigma_q,\ \sigma,\ \tau$) a subset of σ_n; the upper index denotes the number of the subset, the lower one the number of variables constituting this subset; $\overline{\sigma}$ the set of variables complementing σ to σ_n;

$$\chi(\sigma_q^i) = \chi_{i_1}(x_{i_1})\chi_{i_2}(x_{i_2}) \cdots \chi_{i_q}(x_{i_q}), \qquad \sigma_q^i = \{x_{i_1}, \dots, x_{i_q}\};$$

$$\varphi(\sigma_n) = \varphi(x_1, x_2, \dots, x_n), \qquad f(\sigma_q^i) = f(x_{i_1}, x_{i_2}, \dots, x_{i_q})$$

and so on;

$$\int_{D_{\sigma_q}} f(\sigma_p)\, d\chi(\sigma_q)$$

is the Lebesgue-Stieltjes integral of $f(\sigma_p)$ over the section

$$D_{\sigma_q} = \{a_i \leqslant x_i \leqslant b_i,\ x_i \in \sigma_q\}$$

of the parallelepiped D by the corresponding q-dimensional hyperplane; in what follows such integration will be called the averaging of the function $f(\sigma_p)$ over the set of variables σ_q.

Lemma 1 and Theorem 2 are proved on the basis of the following obvious properties of the averaging operation:

1.
$$f(\sigma) = \int_{D_{\overline{\sigma}}} f(\sigma)\, d\chi(\overline{\sigma}).$$

2.
$$\int_{D_\sigma} f(\tau)\, d\chi(\sigma) = \int_{D_{\sigma^2}} \left\{ \int_{D_{\sigma^1}} f(\tau)\, d\chi(\sigma^1) \right\} d\chi(\sigma^2), \qquad \sigma^1 \cup \sigma^2 = \sigma.$$

3.
$$\int_{D_\sigma} \sum_{i=1}^{k} f_i(\tau^i)\, d\chi(\sigma) = \sum_{i=1}^{k} \int_{D_\sigma} f_i(\tau^i)\, d\chi(\sigma), \qquad k < \infty.$$

4.
$$\int_{D_\sigma} f(\tau)\, d\chi(\tau) \geqslant 0, \quad \text{if } f(\tau) \geqslant 0^{**}.$$

Lemma 1. *If the function $\varphi(\sigma_n)$ in the parallelepiped D is a sum of functions of sets of variables $\sigma^1, \sigma^2, \dots, \sigma^m$ that do not contain one another:*

$$\varphi(\sigma_n) = \sum_{i=1}^{m} f_i(\sigma^i), \qquad \sigma^i \not\subseteq \sigma^j \text{ for } i \neq j; \quad 1 \leqslant i, j \leqslant m,$$

* a_i, b_i are not necessarily finite.
** The dot above the signs $\geqslant$ and $=$ denotes almost everywhere with respect to the measure χ.

then

$$\varphi(\sigma_n) = \sum_{1 \le i \le m} \varphi(\sigma^i) - \sum_{1 \le i < j \le m} \varphi(\sigma^i \cap \sigma^j) + \dots + (-1)^{m+1} \varphi(\sigma^1 \cap \sigma^2 \cap \dots \cap \sigma^m),$$

where by $\varphi(\sigma^i)$, $\varphi(\sigma^i \cap \sigma^j)$, etc., are denoted the averages of $\varphi(\sigma_n)$ with respect to the corresponding sets of variables σ^i, $\sigma^i \cap \sigma^j$, etc.

Theorem 1. The best mean-square approximation in D, with respect to the measure $\chi(\sigma_n)$, of the function $\varphi(\sigma_n)$, for prescribed combinations of variables $\sigma^1, \sigma^2, \dots, \sigma^m$ in the approximating sum, is given by the function

$$\psi(\sigma_n) = \sum_{1\le i\le m} \varphi(\sigma^i) - \sum_{1\le i<j\le m} \varphi(\sigma^i \cap \sigma^j) + ... + (-1)^{m+1}\varphi(\sigma^1 \cap \sigma^2 \cap ... \cap \sigma^m),$$

and, moreover,

$$\int_D \{\varphi(\sigma_n) - \psi(\sigma_n)\}^2 \, d\chi(\sigma_n) = \int_D \varphi^2(\sigma_n)\, d\chi(\sigma_n) - \int_D \psi^2(\sigma_n)\, d\chi(\sigma_n).$$

The results obtained may be used for the analysis and approximate representation of functions of many variables arising in applications.

Recently a number of interesting works of a theoretical-functional nature have been published, for example ([1–3]), concerning the problem of approximating functions of many variables by sums of functions of a smaller number of variables, in which the existence and uniqueness of the best approximating function are proved.

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CITED LITERATURE

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[3] M.-B. A. Babaev, *Dokl. AN AzerbSSR*, **23**, No. 1, 3 (1967).

Note: Figure translations are in progress. See original paper for figures.