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# LAMÉ PROBLEMS IN THE GRADIENT THEORY OF ELASTICITY

1968

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## Abstract

## Full Text

UDC 539.37

### THEORY OF ELASTICITY

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# LAMÉ PROBLEMS IN THE GRADIENT THEORY OF ELASTICITY

*(Presented by Academician L. I. Sedov, 2 XII 1967)*

The fundamental equations of the theory under consideration, describing media whose characteristics include, in addition to strains, also strain gradients, are contained in works ([1]-[5]). The equations of the gradient theory of elasticity are of higher order than the corresponding equations of the classical theory and require additional boundary conditions for their solution. Such conditions were obtained in ([4], [5]); moreover, in the author' s work ([5]) they were considered as a special case of conditions on discontinuities, namely as conditions on the interface between media. The Lamé problems solved within the framework of the gradient theory reveal the features of these conditions and make it possible to estimate the influence of higher derivatives.

Let us consider an isotropic continuous medium whose free energy $F$ depends both on the strains $\varepsilon_{ij} = {}^1/_2(\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ and on the strain gradients $\varepsilon_{ij,k} \equiv \nabla_k \varepsilon_{ij}$, and is represented by a positive definite quadratic form of its arguments

$$F = \frac{\lambda}{2}\varepsilon_{ii}^2 + \mu\varepsilon_{ij}\varepsilon_{ij} + k_1\varepsilon_{ij,k}\varepsilon_{ij,k} + k_2\varepsilon_{ij,k}\varepsilon_{ik,j} + k_3\varepsilon_{ij,i}\varepsilon_{kj,k} +$$

$$+k_4\varepsilon_{ij,i}\varepsilon_{kk,j} + k_5\varepsilon_{ii,j}\varepsilon_{kk,j}.$$

Here $\lambda$ and $\mu$ are the Lamé coefficients; $k_1, k_2, k_3, k_4, k_5$ are new elastic constants*. The stresses in such a medium are given by the formulas:

$$P_{ij}/\rho_0 = \partial F/\partial\varepsilon_{ij} - \nabla_k Q_{kij}; \qquad Q_{kij} = \partial F/\partial\varepsilon_{ij,k}. \tag{1}$$

In expanded form the stress tensor has the form

$$\frac{1}{\rho_0}P_{\alpha\alpha} = \lambda\theta + 2\mu\frac{\partial u_\alpha}{\partial x_\alpha} - \left[2\tau_1\nabla^2\frac{\partial u_\alpha}{\partial x_\alpha} + \tau_2\frac{\partial^2\theta}{\partial x_\alpha^2} + \tau_3\nabla^2\theta\right],$$

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$$\frac{1}{\rho_0}P_{\alpha\beta} = \mu\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right) - \left[\tau_1\nabla^2\left(\frac{\partial u_\alpha}{\partial x_\beta} + \frac{\partial u_\beta}{\partial x_\alpha}\right) + \tau_2\frac{\partial^2\theta}{\partial x_\alpha \partial x_\beta}\right], \qquad \alpha \neq \beta, \tag{2}$$

$$\theta = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}.$$

Here $2\tau_1 = 2k_1 + k_2 + k_3$, $\tau_2 = k_2 + k_3 + k_4$, $\tau_3 = k_4 + 2k_5$.

From the equilibrium equations $\nabla_k P_{ik} = 0$ we obtain a generalization of the Lamé equations for the case of the media considered:

$$(\lambda + \mu)\frac{\partial\theta}{\partial x_i} + \mu\nabla^2 u_i - \nabla^2\left(A\frac{\partial\theta}{\partial x_i} + B\nabla^2 u_i\right) = 0. \tag{3}$$

Here $A = k_1 + {}^3/_2 k_2 + {}^3/_2 k_3 + 2k_4 + 2k_5$, $B = k_1 + {}^1/_2 k_2 + {}^1/_2 k_3$.

For the Lamé problems of an infinitely long tube and a spherical shell subjected to uniform external and internal pressure and bordering a medium whose characteristics include strain gradients—

* These constants can either be found by processing experimental data or expressed in terms of the molecular parameters of the medium by means of various statistical theories.

deformations are absent, the solution is sought in the form

$$u_i = \varphi(r)x_i, \qquad r = x_k x_k. \tag{4}$$

The index $i = 1, 2$ for the tube (with $u_3 = 0$); $i = 1, 2, 3$ for the spherical shell. In both cases equations (3) are easily integrated, and the solution is given by the formula

$$\varphi(r) = c_1 + c_2 r^{-m} + c_3 K_m\left(i\sqrt{\frac{\lambda + 2\mu}{A + B}}\, r\right) + c_4 L_m\left(i\sqrt{\frac{\lambda + 2\mu}{A + B}}\, r\right),$$
$$K_m = I_{m/2}\left(i\sqrt{\frac{\lambda + 2\mu}{A + B}}\, r\right) \Big/ \sqrt{\frac{\lambda + 2\mu}{A + B}}\, r^{-m/2}, \tag{5}$$
$$L_m = Y_{m/2}\left(i\sqrt{\frac{\lambda + 2\mu}{A + B}}\, r\right) \Big/ \sqrt{\frac{\lambda + 2\mu}{A + B}}\, r^{-m/2}.$$

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Here $I_{m/2}$, $Y_{m/2}$ are Bessel functions of purely imaginary argument of order $m/2$. $m = 2$ for the tube, $m = 3$ for the spherical shell. In the latter case the Bessel functions are expressed in terms of elementary transcendental functions.

To determine the four constants of integration we write down the boundary conditions ([5])

$$P_{jk}n_j - D_i(Q_{sik}n_s)\big|_{r=a,b} = \begin{cases} -P_a n_k & \text{for } r = a, \\ -P_b n_k & \text{for } r = b, \end{cases} \qquad Q_{sik}n_s n_i\big|_{r=a,b} = 0. \quad (6)$$

Here $a$ and $b$ are, respectively, the inner and outer radii of the shells; $n_i$ is the unit vector of the outward normal to the boundary surface; $D_i$ is the differential operator ([4,5]) denoting the derivative "along the surface." The first relation in (6) shows that the force acting on a surface element is composed of two parts: the ordinary one, associated with the stresses $P_{jk}n_j$, and an additional one arising from the nonuniform distribution of the tensor $Q_{sik}$ over the surface and from the curvature of the surface itself. The second relation in (6), which arises only in the presence of higher derivatives, shows that the vectors (with respect to the index $i$ or $k$) $Q_{sik}n_s$ must lie in the plane tangent to the boundary. Substituting the solution (5) into formulas (6) and carrying out the simple but rather cumbersome procedure of determining the constants of integration, we obtain for them the expressions

$$\begin{gathered} c_1 = \frac{a^m P_a - b^m P_b}{(m\lambda + 2\mu)(b^m - a^m)}\left[1 - \frac{(P_b - P_a)(m\lambda + 2\mu)}{a^m P_a - b^m P_b}\,\sigma_m \tau_m\right], \\ c_2 = \frac{(P_b - P_a)a^m b^m}{2\mu(m-1)(b^m - a^m)}\left[1 + (m\lambda + 2\mu)\rho_m \tau_m\right], \\ c_3 = -(P_b - P_a)\tau_m; \qquad c_4 = -(P_b - P_a)\nu_m \tau_m. \end{gathered} \quad (7)$$

Here the following notation has been introduced:

$$\begin{gathered} \nu_m = \frac{(\lambda + 2\mu)[K_m(b)b^{m+2} - K_m(a)a^{m+2}] - 2(m-1)T[K'_m(b)b^{m+1} - K'_m(a)a^{m+1}]}{(\lambda + 2\mu)[L_m(b)b^{m+2} - L_m(a)a^{m+2}] - 2(m-1)T[L'_m(b)b^{m+1} - L'_m(a)a^{m+1}]}, \\ \pi_m = [K_m(a) - \nu_m L_m(a)] - \frac{2(m-1)T}{a(\lambda + 2\mu)}[K'_m(a) - \nu_m L'_m(a)], \\ \rho_m = [K_m(b) - K_m(a)] - \nu_m[L_m(b) - L_m(a)], \\ \sigma_m = [b^m K_m(b) - a^m K_m(a)] - \nu_m[b^m L_m(b) - a^m L_m(a)], \\ \tau_m = \frac{2mTb^m}{a^2(\lambda + 2\mu)} \Big/ \left[2\mu(b^m - a^m)\pi_m - \frac{2m(m\lambda + 2\mu)T}{a^2(\lambda + 2\mu)}b^m \rho_m\right], \\ T = k_1 + k_2. \end{gathered} \quad (8)$$

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The expressions preceding the square brackets for $c_1$ and $c_2$ are the corresponding coefficients in the classical Lamé problems.

To estimate the contribution made by the higher derivatives, let us consider the case of a thin spherical shell. In the case of a cylindrical shell the results are analogous. Assuming, for example, $T \simeq A+B$ and introducing the dimensionless parameter $S = \sqrt{\lambda + 2\mu / A + B}\, R$, we obtain, to within small quantities of higher order with respect to the quantity $\delta/R$ ($\delta = b - a$ is the thickness of the shell, assumed to be much smaller than its radius), the following formulas for comparing the classical ($\hat{c}$) and the newly obtained ($c$) solutions of the problem:

$$\frac{c_1 - \hat{c}_1}{\hat{c}_1} = -\frac{9\lambda + 6\mu}{(9\lambda + 8\mu) + 2\mu S^2}; \qquad \frac{c_2 - \hat{c}_2}{\hat{c}_2} = -\frac{9\lambda + 6\mu}{(9\lambda + 8\mu) + 2\mu S^2}. \tag{9}$$

Comparison of the quantities entering the last two terms of solution (5) with the first terms is made according to their contribution to the deformation $\varepsilon_{\varphi\varphi} = u/r = \varphi(r)$,

$$\frac{c_3 e^s \left(s^{-2} - s^{-3}\right) + c_4 e^{-s} \left(s^{-2} + s^{-3}\right) - \left(\hat{c}_1 + \hat{c}_2 R^{-3}\right)}{\hat{c}_1 + \hat{c}_2 R^{-3}} = \frac{12\mu}{(9\lambda + 8\mu) + 2\mu s^2} - 1. \tag{10}$$

It follows from formulas (9)–(10) that as $s \to \infty$ the solution of the problem passes over into the classical one; moreover, such a situation may occur either in the case $A + B \to 0$, i.e., in the case of absence of higher derivatives, or when $R \to \infty$, i.e., for sufficiently small curvature of the shell. Conversely, as $s \to 0$, the corrections associated with the presence of strain gradients become significant. For the radius of curvature of the shell this gives the condition

$$R \ll \sqrt{\frac{A + B}{\lambda + 2\mu}}. \tag{11}$$

Thus, formula (11) makes it possible to estimate the minimum curvature of the boundary surface at which strain gradients substantially affect the solution. It should be noted that the formulas obtained, in the limiting case $b \to \infty$, give the solutions for a plane and for space weakened respectively by a circular and a spherical hole. In both cases, as also in the couple-stress theory, one may note a reduction in stress concentration.

The author expresses gratitude to L. I. Sedov for his interest in the work and for valuable suggestions.

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Received
11 XI 1967

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*Note: Figure translations are in progress. See original paper for figures.*

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