THE FOURIER TRANSFORM METHOD FOR EQUATIONS IN FUNCTIONAL DERIVATIVES

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Abstract

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THE FOURIER TRANSFORM METHOD FOR EQUATIONS IN FUNCTIONAL DERIVATIVES

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1. Recently a number of works have appeared devoted to infinite-dimensional analogues of parabolic equations of the form

$$\partial u/\partial t = Bu, \tag{1}$$

where B is a certain elliptic operator of second order (see, for example, ([1]) and the literature cited there). The usual approach here is based on approximating equations in functional derivatives by the corresponding equations for functions of a finite number of variables. Here we shall consider another approach to the same problem, based on the application of Fourier transforms to equations in functional derivatives.

Consider a rigged Hilbert space

$$\Phi \subset H \subset \Phi'$$

(see ([2])), and let $\mu(M)$ be a charge in this space, defined on the σ-algebra Z generated by cylindrical sets, σ-additive in Φ' and differentiable in all directions from Φ (see ([3])). This last condition means that for every $y \in \Phi$ the equality

$$\mu(M + \varepsilon y) = \mu(M) + \varepsilon(\mu'(M), y) + \varepsilon^2(\mu''(M)y, y) + o(\varepsilon^2), \tag{2}$$

holds. Here $\mu'(M)$, for each fixed M, is an element of Φ', while $\mu''(M)$ is an operator acting from Φ into Φ'. For fixed $y \in \Phi$, each of the terms occurring in (2) is a charge defined on the σ-algebra Z.

Define the Laplace operator of μ as the trace of the operator $\mu''(M)$:

$$\Delta\mu(M) = \operatorname{sp} \mu''(M) \tag{3}$$

(if the charge μ is such that this expression has meaning).

2. We shall now consider the equation

$$\partial\mu/\partial t = \Delta\mu, \tag{4}$$

regarding the unknown μ not as a functional, as is usually done, but as a charge depending on t ($0 \le t < \infty$), σ-additive in Φ' and such that the Laplace operator is defined for it. This point of view has not only certain analytic advantages, connected with the fact that in an infinite-dimensional space the Fourier transform for charges has a perfectly clear meaning, whereas for functionals there is no corresponding convenient concept, but also appears to correspond more closely to the physical and probabilistic meaning of equations of type (1). Indeed, such an equation describes a certain Markov process for which Φ' serves as the state space, and $\mu = \mu(t, M)$ for each t gives the corresponding probability distribution in Φ'. If the state space is finite-dimensional, then it is natural to consider, instead of the distribution $\mu(t, M)$, its density $u(t, \varphi)$. However, in the infinite-dimensional Φ' there is no measure,

analogous to Lebesgue measure in R_n, which it would be natural to take as fundamental and with respect to which one would take the density of the distribution.

3. Thus, we shall consider the following Cauchy problem: an equation (4) is given, and the initial condition

$$\mu(0, M) = \mu_0(M), \tag{5}$$

where $\mu_0(M)$ is a charge in Φ having finite variation.

If $\nu(M)$ is some charge in Φ', then its Fourier transform is defined (see, for example, (2))

$$\tilde{\nu}(\varphi) \int_{\Phi'} e^{i(\varphi,\psi)}\, d\nu(\psi),$$

and is a continuous, and in the case of positivity of ν also a positive definite functional on Φ.

The following fact holds, analogous to the known properties of the Fourier transform for functions of a finite number of variables: if $\nu(\varphi)$ is the Fourier transform of some charge ν, for which the Laplace operator exists, then the Fourier transform of the charge $\Delta\nu$ is $-(\varphi, \varphi)\tilde{\nu}(\varphi)$.

This circumstance makes it possible to apply, for the solution of the Cauchy problem (4), (5), essentially the very same Fourier transform method as in the finite-dimensional case. The Fourier transform takes equation (4) into the ordinary equation

$$d\tilde{\mu}/dt = -(\varphi,\varphi)\tilde{\mu}, \tag{6}$$

and the initial condition (5) into the condition

$$\tilde{\mu}(0,\varphi) = \tilde{\mu}_0(\varphi). \tag{7}$$

Hence we obtain that

$$\tilde{\mu}(t,\varphi) = e^{-(\varphi,\varphi)t}\tilde{\mu}_0(\varphi). \tag{8}$$

It remains to find a charge, depending on t, for which (8) serves as the Fourier transform. But $e^{-(\varphi,\varphi)t}$ is the Fourier transform of the Gaussian measure ω_t in the space Φ', corresponding to the quadratic form $(\varphi,\varphi)t$, while $\tilde{\mu}_0(\varphi)$ is the Fourier transform of the "initial" charge μ_0. Since multiplication of characteristic functionals corresponds to convolution of charges, from (8) we obtain:

$$\mu(t,M) = \int_{\Gamma'} \mu_0(M-x)\, d\omega_t(x). \tag{9}$$

4. The very same arguments can also be applied to the equation of a somewhat more general form

$$\partial\mu/\partial t = B\mu, \tag{10}$$

where B is an elliptic operator, i.e.

$$B\mu(M) = \mathrm{sp}\, A^*\mu''(M)A. \tag{11}$$

(Here A is an operator chosen so that expression (11) has meaning. For example, if the operator $\mu''(M)$ is defined on some Φ_n—the completion of Φ with respect to the norm $\|\ \|_n$, then A maps Φ into Φ_n.) The solution of the Cauchy problem for this equation is given by the same formula (9), but only with the "standard" Gaussian measure ω_t replaced by the Gaussian measure $\omega_{A,t}$, determined by the quadratic form $(A\varphi, A\varphi)t$.

Thus, considering the desired function in the parabolic equation (4) or (10) as a σ-additive function of a set, and not as a function of a point, to a considerable extent removes the difficulties associated with the infinite-dimensionality of the argument space and makes it possible, essentially without changes, to transfer the Fourier transform method from equations with partial derivatives to the case considered by us.

5. The theorem on the existence and uniqueness of the solution of the Cauchy problem for equations of the form (10) is found in the work of Yu. L. Daletskii ([1]). The method of proof used there is based on considering the corresponding finite-dimensional approximations and justifying the passage to the limit. This method imposes certain smoothness requirements on the initial conditions. The Fourier transform method does not require smoothness of the initial conditions, but it does require the assumption that the variation of the charge μ_0 is finite. The solution itself of the Cauchy problem for equation (4) or (10) is likewise sought in the class of charges satisfying this condition for each t. To apply the Fourier method in a more general case, it is necessary to develop an appropriate version of the theory of generalized functions. Apparently, such a theory can be developed quite far and consistently.

The considerations set forth above can evidently be extended also to equations of a type different from equation (1). Some preliminary considerations connected with the construction of such a theory are presented in ([4]).

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CITED LITERATURE

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[2] I. M. Gel' fand, N. Ya. Vilenkin, *Generalized Functions*, vol. 4, Moscow, 1961.
[3] S. V. Fomin, UMN, **23**, no. 1 (139) (1968).
[4] S. V. Fomin, UMN, **23**, no. 2 (140) (1968).

Note: Figure translations are in progress. See original paper for figures.