ON CONVERGENT OPERATORS

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Abstract

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MATHEMATICS

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This note gives one result (Theorem 2) on convergent operators, closely connected with the question of convergence of projection methods. Some estimates of convergence of the Bubnov–Galerkin method are indicated.

1. Let L be a linear (in general, unbounded) operator with domain of definition $D(L)$ in a Banach space $\mathfrak{E}$ and range $R(L)$ in a Banach space E. The projection method ([1]) for solving the equation $Lu = f$ consists of the following. Two systems of subspaces $\{\mathfrak{E}_\lambda\}_{\lambda\in\Lambda}$ and $\{E_\lambda\}_{\lambda\in\Lambda}$ are given,

$$\mathfrak{E}_\lambda \subset D(L) \subset \mathfrak{E}, \qquad E_\lambda \subset E \qquad (\lambda \in \Lambda),$$

where Λ is some set; we shall assume that Λ is a subset of the number line for which $+\infty$ is a limit point. Further, projectors $P_\lambda (\lambda \in \Lambda)$ are given, projecting E onto E_λ. The equation $Lu = f$ is then replaced by the approximate one

$$P_\lambda(Lu_\lambda - f) = 0 \qquad (u_\lambda \in \mathfrak{E}_\lambda), \tag{1}$$

the last parentheses meaning that the solution is sought in $\mathfrak{E}_\lambda$.

N. I. Pol' skii ([1]) indicated conditions for convergence of the projection method (1) in the case when the subspaces $\mathfrak{E}_\lambda$ and E_λ are finite-dimensional. The following theorem is a simple generalization of N. I. Pol' skii' s result to the case when the subspaces are not necessarily finite-dimensional.

Theorem 1. *Let $\overline{R(L)} = E$, and let L be invertible on $R(L)$. Let the subspaces E_λ and $F_\lambda = L\mathfrak{E}_\lambda$ $(\lambda \in \Lambda)$ be closed in E, and let the projectors P_λ be uniformly bounded in λ:*

$$\|P_\lambda\| \le \varkappa = \text{const} \qquad (\lambda \in \Lambda).$$

Then, in order that for every $f \in E$ the equation (1), beginning with some $\lambda = \lambda_0$, have a unique solution u_λ, and that as $\lambda \to \infty$ the residual $Lu_\lambda - f$

tend to zero in norm, it is necessary and sufficient that the following conditions be satisfied:

1) the system of subspaces $\{F_\lambda\}_{\lambda\in\Lambda}$ is ultimately dense in E, i.e.

$$\lim \rho(x, E_\lambda) = 0 \quad \text{for every } x \in E;$$

2) for $\lambda \geq \lambda_0$, $\lambda \in \Lambda$, the operator P_λ maps F_λ one-to-one onto E_λ;

3)

$$\lim_{\lambda\to\infty} \tau_\lambda > 0, \qquad \text{where} \qquad \tau_\lambda = \inf_{\substack{z\in F_\lambda \\ \|z\|=1}} \|P_\lambda z\|.$$

The rate of convergence, under conditions 1)–3), is characterized by the inequality

$$\rho(f, F_\lambda) \leq \|Lu_\lambda - f\| \leq (1 + \varkappa/\tau_\lambda)\rho(f, F_\lambda).$$

Let us pass to the consideration of the case where the space $E = H$ is Hilbert and P_λ $(\lambda \in \Lambda)$ are orthoprojectors, $P_\lambda H = H_\lambda = E_\lambda$. The quantity

$$\theta(F_\lambda, H_\lambda) = \max\left\{ \sup_{x\in F_\lambda,\ \|x\|=1} \rho(x, H_\lambda), \qquad \sup_{x\in H_\lambda,\ \|x\|=1} \rho(x, F_\lambda) \right\}$$

is called the gap ([2]) between the subspaces F_λ and H_λ. In the case when F_λ and H_λ are finite-dimensional, the following assertion is quite obvious and was noted in ([1]).

Lemma. The orthoprojector P_λ $(P_\lambda H = H_\lambda)$ maps a closed subspace $F_\lambda \subset H$ one-to-one onto a closed subspace $H_\lambda \subset H$ if and only if $\theta_\lambda = \theta(F_\lambda, H_\lambda) < 1$. In this case the numbers θ_λ and τ_λ (see Theorem 1) are related by

$$\theta_\lambda^2 + \tau_\lambda^2 = 1.$$

Taking this lemma into account, Theorem 1 can be reformulated in the following way (the convergence estimate is sharpened):

Theorem 1′. Let a linear operator L, acting from a Banach space $\mathfrak{E}$ into a Hilbert space H, have range $R(L)$ dense in H and be invertible on $R(L)$. Let the subspaces H_λ and

$$F_\lambda = L\mathfrak{E}_\lambda \quad (\lambda \in \Lambda)$$

be closed in H, and let P_λ and Π_λ be the corresponding orthoprojectors.

Then, in order that for every $f \in H$ equation (1), starting with some λ, have a unique solution u_λ, and that as $\lambda \to \infty$ the residual $Lu_\lambda - f$ tend to zero in norm, it is necessary and sufficient that the following conditions hold:

1′) the system of subspaces $\{F_\lambda\}_{\lambda \in \Lambda}$ is limitingly dense in H;

2′) $\overline{\lim}_{\lambda\to\infty} \theta_\lambda < 1$, where $\theta_\lambda = \theta(F_\lambda, H_\lambda)$ is the aperture of the subspaces F_λ and H_λ.

The rate of convergence under conditions 1′ and 2′ is characterized by the inequality

$$\|f - \Pi_\lambda f\| \leqslant \|Lu_\lambda - f\| \leqslant \frac{1}{\sqrt{1-\theta_\lambda^2}} \|f - \Pi_\lambda f\|.$$

Note that conditions 1′ and 2′ do not imply the limiting density of the system $\{H_\lambda\}_{\lambda\in\Lambda}$ in H.

2. Theorem 2. Let A and B be unbounded self-adjoint positive definite operators in a Hilbert space H, having a common domain of definition* $D(A) = D(B)$. Let P_λ, $0 < \lambda < \infty$, be the resolution of the identity (see, for example, ([4])) corresponding to the operator B,

$$B = \int_0^\infty \lambda \, dP_\lambda,$$

and let $H_\lambda = P_\lambda H$.

Then for arbitrary α and β, $0 \leqslant \alpha, \beta \leqslant 1$, the relation

$$\overline{\lim}_{\lambda\to\infty} \theta(A^\alpha H_\lambda, A^\beta H_\lambda) < 1$$

holds.

From the similarity of the operators A and B according to the theorem of E. Heinz (see, for example, ([5])) it follows that $D(A^\alpha) = D(B^\alpha)$ $(0 \leqslant \alpha \leqslant 1)$, and this entails the boundedness of the operators $A^\alpha B^{-\alpha}$ and

$$(A^\alpha B^{-\alpha})^{-1} = B^\alpha A^{-\alpha} \quad (0 \leqslant \alpha \leqslant 1).$$

The subspaces

$$A^\alpha H_\lambda = A^\alpha B^{-\alpha} H_\lambda \quad (0 \leqslant \alpha \leqslant 1)$$

are closed in H, and the system of subspaces

$$\{A^\alpha H_\lambda\}_{\lambda \in (0,\infty)}$$

for $0 \leqslant \alpha \leqslant 1$ is limitingly dense in H.

The idea of the proof of Theorem 2 is as follows. One considers the equation $A^\alpha u = g$ and the projection method

$$P_\lambda^{(\beta)}(A^\alpha u_\lambda - g) = 0 \quad (u_\lambda \in H_\lambda)$$

for its solution; here $P_\lambda^{(\beta)}$ is the orthoprojector corresponding to the subspace $A^\beta H_\lambda$. It is proved that the residual $A^\alpha u_\lambda - g$ tends to zero as $\lambda \to \infty$ for every $g \in H$, and this, by virtue of Theorem 1, is equivalent to the assertion of Theorem 2.

3. As an application, we indicate one scale of convergence estimates for the Bubnov–Galerkin method. Let the operators A and B satisfy the conditions of Theorem 2. Consider the equation

$$Lu \equiv Au + Ku = f, \tag{2}$$

where K is a linear operator in H with domain of definition $D(K) \supset D(A)$. An approximate solution of equation (2) is found by the Bubnov–Galerkin method from the conditions

$$P_\lambda(Lu_\lambda - f) = 0 \quad (u_\lambda \in H_\lambda), \tag{3}$$

where P_λ, $0 < \lambda < \infty$, is the resolution of the identity corresponding to the operator B, and $H_\lambda = P_\lambda H$.

* Such operators are called similar (see, for example, ([3])).

Suppose that for some α_0 ($0 \le \alpha_0 \le 1$) the operator $A^{\alpha_0 - 1} K A^{-\alpha_0}$ is completely continuous in H, and that the homogeneous equation $x + T_{\alpha_0} x = 0$, where T_{α_0} is the continuous extension of the operator $A^{\alpha_0-1} K A^{-\alpha_0}$ to H, has only the zero solution. Then the equation $x + T_{\alpha_0} x = A^{\alpha_0 - 1} f$ has a unique solution x^*, and the element $u^* = A^{-\alpha_0} x^* \in D(A^{\alpha_0})$ is taken as the (generalized, if $u^* \notin D(A)$) solution of equation (2). Let* $u^* \in D(B^{\alpha_1})$; obviously, $\alpha_1 \ge \alpha_0$.

Under the assumptions made, there exists such a λ_0 that for $\lambda \ge \lambda_0$ there is a unique approximation u_λ satisfying conditions (3), and for $\alpha \le \alpha_1$ the convergence $\|B^\alpha(u_\lambda - u^*)\| \to 0$ ($\lambda \to \infty$) holds, with the estimate

$$\varepsilon_\lambda^{(\alpha)} \ll \|B^\alpha(u_\lambda - u^*)\| < c\varepsilon_\lambda^{(\alpha)} \qquad (\alpha_0 \le \alpha \le \alpha_1), \tag{4}$$

where the constant c does not depend on λ, α, and f, and

$$\varepsilon_\lambda^{(\alpha)} = \|B^\alpha u^* - P_\lambda B^\alpha u^*\| = o(\lambda^{\alpha - \alpha_1}).$$

The proof is based on Theorems 1′ and 2. In some particular cases (for $\alpha = 0$, $\alpha = 1/2$, and $\alpha = 1$) the estimate (4) was established earlier by G. M. Vainikko ([6]); similar results are due to A. V. Dzhishkariani ([7]).

Remark. It is not hard to see that if α_0 and α_0' are two values of the parameter α for which the operator $A^{\alpha-1} K A^{-\alpha}$ is completely continuous, then invertibility of one of the operators $I + T_{\alpha_0}$ and $I + T_{\alpha_0'}$ entails invertibility of the other. Both these values of the parameter α lead, in the manner indicated above, to one and the same solution u^* of equation (2). If, in particular, the operator KA^{-1} is completely continuous in H, then necessarily $u^* \in D(A)$, and in the estimates (4) one may put $\alpha_1 = 1$.

Finally, let us consider the case when the operator B^{-1} is completely continuous. Then the spectrum of the operator B consists of a sequence of positive eigenvalues λ_k, accumulating only at $+\infty$. Let φ_k be the eigenvector of B corresponding to λ_k:

$$B\varphi_k = \lambda_k \varphi_k \qquad (0 < \lambda_1 \leq \lambda_2 \leq \dots;\ \lambda_k \to \infty \text{ as } k \to \infty).$$

The subspace H_λ coincides with the linear span of those elements φ_k $(k = 1, 2, \dots)$ for which $\lambda_k \leq \lambda$. Conditions (3) are written in the form

$$(Lu_n - f, \varphi_i) = 0 \qquad \left(u_n = \sum_{k=1}^{n} \xi_k \varphi_k;\ i = 1, 2, \dots, n\right) \tag{3'}$$

(the Bubnov–Galerkin system of equations). The estimates (4) must be modified accordingly. We obtain, in particular, that

$$\|B^\alpha(u_n - u^*)\| = o(\lambda_n^{\alpha - \alpha_1}) \qquad (\alpha_0 \leq \alpha \leq \alpha_1). \tag{4'}$$

In the case when $K = 0$, the Bubnov–Galerkin method $(3')$ becomes the Ritz method. In the estimates $(4')$ one may put $\alpha_0 = 0$, $\alpha_1 = 1$. From the estimate $(4')$ for $\alpha = \alpha_1 = 1$ it follows that, as $n \to \infty$, the residual $Au_n - f$ tends to zero (for any $f \in H$). This remarkable property of the coordinate sequence $\{\varphi_k\}$ under consideration was discovered by S. G. Mikhlin ([8]); in ([3]) a proof is given under some additional restrictions.

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* The case $\alpha_1 > 1$ is not excluded.

Note: Figure translations are in progress. See original paper for figures.