ON SECTIONS OVER ZERO-DIMENSIONAL SUBSETS OF QUOTIENT SPACES OF LOCALLY BICOMPACT GROUPS

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Abstract

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ON SECTIONS OVER ZERO-DIMENSIONAL SUBSETS OF QUOTIENT SPACES OF LOCALLY BICOMPACT GROUPS

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Reiter in (1) proved that if in the quotient group G/H of a locally bicompact group G with the first axiom of countability one takes an arbitrary derived (i.e., containing no nonempty perfect subset) closed set A', then in G there exists a closed set A which is mapped onto A' homeomorphically by means of the natural mapping $G \to G/H$. In the same paper (1) Reiter posed the question of extending his result to arbitrary locally bicompact groups. It turns out that such an extension is possible.

Theorem 1. *If G is a locally bicompact group and H is its closed subgroup, then for every paracompact zero-dimensional* set $A' \subseteq G/H = X$ there is a set $A \subseteq G$ which is mapped onto A' homeomorphically by means of the natural mapping $p : G \to X$.*

Remark. As A' one may take, for example, any countable subset of X or any zero-dimensional closed subset of X.

If in the theorem $\dim X < \infty$, then, by Mostert' s theorem (2) (see also (3)), the mapping $p : G \to X$ is a locally trivial fibration. In view of the paracompactness and zero-dimensionality of A', the mapping $p : p^{-1}(A') \to A'$ will already be a trivial fibration, whence the existence of the set A follows.

In the general case we shall need several lemmas.

Lemma 1. *Let X be the limit of such an inverse spectrum*

$$S = \{X_\alpha, \mathfrak{F}_\alpha^\beta\}, \quad \alpha \in \mathfrak{A},$$

that: 1) the indices α are all ordinal numbers less than a certain limit number θ; 2) the projections $\mathfrak{F}_\alpha^{\alpha+1}$ are locally trivial fibrations; 3) for every limit number β, the element of the spectrum X_β is the limit of the spectrum

$$S_\beta = \{X_\alpha, \mathfrak{F}_\alpha^{\alpha'}\}, \quad \alpha < \beta.$$

Then for every paracompact zero-dimensional set $A' \subseteq X_1$ there exists a set A in $X = X_0$ which is mapped onto A' homeomorphically by means of the projection $\mathfrak{F}_1 : X \to X_1$.

Proof. The set $A \subseteq X$ is obtained as the limit of a spectrum $\{A_\alpha, \mathfrak{F}_\alpha^\beta\}$, $\alpha \in \mathfrak{A}$, for which $A_1 = A'$, $A_\alpha \subseteq X_\alpha$, and the mappings $\mathfrak{F}_\alpha^\beta : A_\beta \to A_\alpha$ are homeomorphisms. The sets A_α are constructed by transfinite induction. (The construction of $A_{\alpha+1}$ from A_α uses condition 2) and the fact that the fibration $\mathfrak{F}_\alpha^{\alpha+1} : (\mathfrak{F}_\alpha^{\alpha+1})^{-1} A_\alpha \to A_\alpha$ is trivial. The construction of A_β for a limit β uses condition 3).)

Lemma 2. *If G is a locally bicompact projective-Lie* (^{4}) *group and H is its closed subgroup, then the space of the group G* is the limit of the spectrum $S = \{X_\alpha, \delta_\alpha^\beta\}$, $\alpha \in \mathfrak{A}$, satisfying the conditions of Lemma 1, and $X_1 = G/H$.

Proof. We construct for G an analogue of the Pontryagin-Lee series, beginning with the space G/H. In G one can (^{3}) choose a system of bicompact normal divisors G_α, indexed by all ordinal numbers α, $2 \leq \alpha \leq \omega_\tau$, where τ is the weight of G at a point, such that: 1) $G_{\alpha+1} \subset G_\alpha$, 2) $G_\beta = \bigcap_{\alpha<\beta} G_\alpha$ for limit numbers β; 3) every neighborhood of the identity contains at least one of the normal divisors G_α, in particular $\bigcap_\alpha G_\alpha = e$; 4) the quotient groups G/G_2 and $G_\alpha/G_{\alpha+1}$ are Lee groups.

Denote the subgroups $H \cap G_\alpha$ by H_α, and H by H_1. Since for $\beta > \alpha$ we have the inclusion $H_\beta \subseteq H_\alpha$, the natural projection δ_α^β of the quotient space $X_\beta = G/H_\beta$ onto the quotient space $G/H_\alpha = X_\alpha$ is defined; moreover, for all α the natural projections $\delta_\alpha : G \to X_\alpha$ are defined. Since all subgroups H_α for $\alpha \geq 2$ are bicompact, all projections δ_α and δ_α^β for $\alpha \geq 2$ are perfect.* Thus the spectrum $S = \{X_\alpha, \delta_\alpha^\beta\}$, $1 \leq \alpha < \omega_\tau$, is defined.

Consider a limit number $\beta \leq \omega_\tau$ (putting $G_{\omega_\tau} = e$ and $X_{\omega_\tau} = G$). The space X_β, for each α, $1 \leq \alpha < \beta$, has a continuous mapping δ_α^β onto the space X_α, and $\delta_\alpha^{\alpha'} \cdot \delta_{\alpha'}^\beta = \delta_\alpha^\beta$ for $\alpha' > \alpha$, i.e. there is defined a continuous mapping f_β of the space X_β onto the everywhere dense subset $\bar{X}_\beta$ of the limit of the spectrum $S_\beta = \{X_\alpha, \delta_\alpha^{\alpha'}\}$, $\alpha < \beta$, satisfying the relations $\delta_\alpha^\beta = \pi_\alpha \cdot f_\beta$ (where π_α denotes the projection of the limit of the spectrum S_β onto the element of the spectrum X_α). Since the mappings δ_α^β are bicompact mappings "onto" and $\delta_\alpha^{\alpha'} \cdot \delta_{\alpha'}^\beta = \delta_\alpha^\beta$ for $\alpha' > \alpha$, the mapping f_β is also a bicompact mapping "onto" (if $\delta_\alpha^{\alpha'}(x_{\alpha'}) = x_\alpha$, then $(\delta_{\alpha'}^\beta)^{-1} x_{\alpha'} \subseteq (\delta_\alpha^\beta)^{-1} x_\alpha$). Since $\delta_\alpha^\beta = \pi_\alpha \cdot f_\beta$ and the mapping δ_α^β is perfect, f_β will also be perfect. Finally, let $x_1 \neq x_2 \in X_\beta$, i.e. $x_1 = g_1 H_\beta \neq x_2 = g_2 H_\beta$. If $\delta_\alpha^\beta(x_1) = \delta_\alpha^\beta(x_2)$, then $g_1 H_\alpha = g_2 H_\alpha$, whence follows the existence of an index

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* By dimension is meant dimension defined by means of coverings.

$\alpha_0 < \beta$ such that $g_1 H_{\alpha_0} \neq g_2 H_{\alpha_0}$. (If this were not so, then

$$g_1 H_\beta = g_1 \left(\bigcap_{\alpha<\beta} H_\alpha \right) = \bigcap_{\alpha<\beta} g_1 H_\alpha = \bigcap_{\alpha<\beta} g_2 H_\alpha = g_2 \left(\bigcap_{\alpha<\beta} H_\alpha \right) = g_2 H_\beta,$$

which contradicts the choice of x_1 and x_2.) Thus there exists an index α_0 for which $\delta^\beta_{\alpha_0}(x_1) \neq \delta^\beta_{\alpha_0}(x_2)$. One-to-one-ness, and hence homeomorphism, of f_β is proved.

Finally, let us show that each projection $\delta^{\alpha+1}_\alpha : X_{\alpha+1} \to X_\alpha$ is a locally trivial fibration; for this it suffices to show that any quotient group $H_{\alpha+1}/H_\alpha$ is a manifold ((3), Theorem 14).

The quotient group $H_1/H_2 = H/H \cap G_2$ is homeomorphic to the image of the group H in the quotient group G/G_2, i.e. to a closed subgroup of a Lee group, hence to a Lee group. Indeed, let us put in correspondence to the class $h \cdot H_2 = h(H \cap G_2) \in H_1/H_2$, $h \in H = H_1$, the class $f(hH_2) = hG_2 = h \cdot H_2 G_2 \in G/G_2$. If $h' \cdot H_2 \neq h'' \cdot H_2$, then $h' \cdot G_2 \neq h'' \cdot G_2$, for otherwise $h'g' = h''g''$, g' and $g'' \in G_2$, i.e. $(h'')^{-1} h' = g'' \cdot (g')^{-1} \in H \cap G_2 = H_2$, whence $h' = h''(g''(g')^{-1}) \in h'' \cdot H_2$. Thus the mapping f is one-to-one. It is obviously a mapping "onto." If the set F is closed in H_1/H_2, then its inverse image $\Phi = \Phi \cdot H_2 = \Phi(H \cap G_2)$ is closed in H_1, and then the set $\Phi \cdot G_2$ is closed in G (by virtue of bicompactness of G_2), i.e. so is its image in G/G_2, equal to $f(F)$. The closedness of f is proved.

Let us prove the continuity of f. Let the set F' be closed in the image of the subgroup H in the quotient group G/G_2. Then the inverse image $\widetilde{\Phi}$ of the set is closed in G F', and hence the set $\Phi' = \widetilde{\Phi} \cap H$ is closed in H. The set Φ' coincides with the set $\Phi' \cdot H_2$, since

$$\Phi' = \widetilde{\Phi} \cap H \subseteq (\widetilde{\Phi} \cap H) \cdot H_2 = \Phi' \cdot H_2 = (\widetilde{\Phi} \cap H) \cdot (G_2 \cap H) \subseteq (\widetilde{\Phi} \cdot G_2) \cap H = \widetilde{\Phi} \cap H = \Phi'.$$

The image of the set Φ' in H_1/H_2 is closed and, by the one-to-one character of f, coincides with the set $f^{-1}(F')$. The homeomorphism f is proved.

The space of the factor group $H_\alpha/H_{\alpha+1}$ for $\alpha > 1$ is bicompact and homeomorphic to the image of the group H_α in the Lie group $G_\alpha/G_{\alpha+1}$, i.e. the spaces $H_\alpha/H_{\alpha+1}$ for $\alpha > 1$ are also manifolds. Thus the local triviality of the decompositions

$$\omega^{\alpha+1}_\alpha : X_{\alpha+1} \to X_\alpha$$

in the case of a projective-Lie group G is proved, i.e. the lemma is also proved.

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* That is, they are closed (the image of a closed set is closed) and bicompact (the inverse image of a point is bicompact) mappings.

From Lemmas 1 and 2 the validity of the theorem for projective-Lie groups follows immediately.

If now the group G is arbitrary, then it always contains an open projective-Lie subgroup Γ (^{5}), and the factor space $X = G/H$ decomposes into a sum of pairwise disjoint open sets X_ν, whose inverse images $p^{-1}(X_\nu)$ in G under the natural mapping $p : G \to X$ have, for some $a_\nu \in G$, the form $\Gamma \cdot a_\nu \cdot H$ (see, for example, (^{3}), Lemma 1). The mapping p of the set $\Gamma \cdot a_\nu \subseteq \Gamma \cdot a_\nu \cdot H$ onto X_ν can be identified (^{3}), Lemma 1, with the natural mapping of the projective-Lie group Γ onto the factor space

$$\Gamma/(\Gamma \cap a_\nu \cdot H \cdot a_\nu^{-1}),$$

i.e. everything has been reduced to the case of a projective-Lie group. The theorem is completely proved.

In the case of countable sets, Theorem 1 can be proved under more general assumptions. (All spaces considered are assumed to be completely regular.)

Definition. A space X has the almost everywhere first axiom of countability if X contains an everywhere dense subset of points of countable character in X.

Theorem 2. *Let an open mapping $f : X \to Y$ be given such that every set $f^{-1}(y)$, $y \in Y$, has the almost everywhere first axiom of countability. Then for every σ-discrete (in itself) paracompact set $N \subseteq Y$ there is a set $M \subseteq X$, mapped onto N by f homeomorphically (in short, in X over N there exists a section).*

Corollary 1. *If an open mapping $f : X \to Y$ of a paracompact (^{6}) space X with the first axiom of countability (in particular, complete with the first axiom of countability or perfectly mapped onto a metrizable space with the first axiom of countability) is given, then for every σ-discrete (in itself) paracompact (for example, countable) set $N \subseteq Y$, in X over N there exists a section.*

It is useful to compare this corollary with Theorem 1 from (^{1}).

Corollary 2. *If G is a topological group, H' and H'' are its closed subgroups, $H' \supseteq H''$, and the factor spaces $X' = G/H'$ and $X'' = G/H''$ are metrizable, then in X'', over any σ-discrete set $N \subseteq X'$, with respect to the natural projection $p : X'' \to X'$, there exists a section.*

Corollary 3. *If a group G is almost metrizable (^{7}), and H' and H'' are its closed subgroups, $H' \supseteq H''$, and the factor space H'/H'' is metrizable, then with respect to the natural mapping $p : G/H'' \to G/H'$, for every σ-discrete set closed in G/H' there exists a section in G/H''.*

Theorem 3. *If a group G is almost metrizable and H is its closed subgroup, then with respect to the natural projection $p : G \to G/H$, for every σ-discrete set N closed in G/H, there exists a section in G.*

Instead of closedness of N one may require that N be a paracompact subset of type G_δ in G/H.

The proof of Theorem 3 is analogous to the proof of Theorem 1.

From Theorem 3 follows the validity of Theorem 1, however, only for a σ-discrete set A' closed in X.

Let us also note that spaces X of point-countable type, all points of which are sets of type G_δ in X, satisfy the first axiom of countability; whence it follows

Theorem 4. *The quotient space G/H of an almost metrizable group G is metrizable if and only if at least one of its points has type G_δ in G/H.*

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Note: Figure translations are in progress. See original paper for figures.

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