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Abstract

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V. S. VINOGRADOV

ON AN ANALOGUE OF AN INTEGRAL OF CAUCHY TYPE FOR ANALYTIC FUNCTIONS OF SEVERAL COMPLEX VARIABLES

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Let G be a certain bounded domain in the Euclidean space of real variables x_a, y_a $(a = 1, \dots, n)$, whose boundary is a closed surface S satisfying the Lyapunov conditions. Suppose that a complex-valued function $\varphi(P)$ is given on S, and let us investigate the following problem: under what conditions is the function $\varphi(P)$ the boundary value of some function $u(M)$, analytic in G with respect to the complex variables $z_a = x_a + iy_a$ $(a = 1, \dots, n)$.

This problem was investigated in the works of F. Severi [1,2], when the function $\varphi(P)$ and the surface M depend analytically on local parameters; of F. Fichera [3], when $\varphi(P)$ is the boundary value of some function from $W_2^{(1)}(G)$, and of E. Martinelli [4], when $\varphi(P) \in C^1$.

Our method makes it possible to obtain a necessary and sufficient condition for the case when $\varphi(P)$ satisfies on S only the Hölder condition, and the latter can be expressed in terms of singular integrals.

The investigation is based on the construction of Cauchy integrals and integrals of Cauchy type for the following system of partial differential equations. Namely, consider the system

$$\mathcal{L}u = \sum_{\alpha=1}^{n} \left(\mathcal{P}_\alpha \frac{\partial}{\partial x_\alpha} + \mathcal{Q}_\alpha \frac{\partial}{\partial y_\alpha} \right) u = 0; \tag{1}$$

the matrices $\mathcal{P}_\alpha$ and $\mathcal{Q}_\alpha$ are constructed as follows:

$$\begin{aligned} \mathcal{P}_\alpha &= 1' \times \dots \times 1' \times \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \times 1 \times \dots \times 1, \\ \mathcal{Q}_\alpha &= 1' \times \dots \times 1' \times \begin{Vmatrix} 0 & i \\ -i & 0 \end{Vmatrix} \times 1 \times \dots \times 1. \end{aligned} \tag{2}$$

Here $\times$ denotes the tensor product of matrices, the number of factors is equal to n, and the matrices

$$\begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix} \quad \text{and} \quad \begin{Vmatrix} 0 & i \\ -i & 0 \end{Vmatrix}$$

stand in the α-th place,

$$1 = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \qquad 1' = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}. \tag{3}$$

The matrices (2) arise in constructing a representation of the Clifford algebra; their order is 2^n, and they satisfy the relation

$$\left[\sum_{\alpha=1}^{n} (\mathcal{P}_\alpha \xi_\alpha + \mathcal{Q}_\alpha \eta_\alpha)\right]^2 = \mathcal{E}\left[\sum_{\alpha=1}^{n} (\xi_\alpha^2 + \eta_\alpha^2)\right], \tag{4}$$

where $\mathcal{E}$ is the identity matrix ([5]). From the last condition there follow the ellipticity of the system (1) and the relation

$$\mathcal{L}^2 = \Delta \mathcal{E}, \tag{5}$$

where Δ is the Laplace operator.

We shall now construct for system (1) the Cauchy integral according to the scheme carried out by the author in ([6]).

Let V be a matrix of order 2^n whose elements are complex functions; consider the identity

$$[V\mathcal{L}]\mathbf{u} + V[\mathcal{L}\mathbf{u}] = \sum \left\{ \frac{\partial}{\partial x_\alpha}(V\mathcal{P}_\alpha \mathbf{u}) + \frac{\partial}{\partial y_\alpha}(V\mathcal{Q}_\alpha \mathbf{u}) \right\}. \tag{6}$$

Here $[V\mathcal{L}]$ means that the differential operator is applied only to the matrix V (on the left), while $[\mathcal{L}\mathbf{u}]$ denotes the application of the operator $\mathcal{L}$ only to the vector $\mathbf{u}$.

Putting

$$V = \left[\frac{1}{r^{2n-2}(M_0, M)}\, \mathcal{C} \cdot \mathcal{L}\right],$$

where $r(M_0, M)$ is the distance between the points M_0 and M; integrating identity (6) over the domain G and applying the Gauss–Ostrogradsky formula, we obtain an expression for $\mathbf{u}(M_0)$

$$\mathbf{u}(M_0) = K\int_S \left[\frac{1}{r^{2n-2}(M_0,M)}\,\mathcal{L}\right]\left[\sum \alpha_i \mathcal{P}_i + \beta_i \mathcal{Q}_i\right]\mathbf{u}(M)\,dS + \int_G \left[\frac{1}{r^{2n-2}(M_0,M)}\,\mathcal{L}\right][\mathcal{L}\mathbf{u}]\,dv, \tag{7}$$

where $\alpha_i = \cos\widehat{nx_i}$, $\beta_i = \cos\widehat{ny_i}$; dS is the surface element; dv is the volume element; $K = (-1)/(2n-2)S_{2n}$; S_{2n} is the area of the surface of the unit sphere.

The order of the singularity

$$\left[\frac{1}{r^{2n-2}(M_0,M)}\,\mathcal{L}\right]$$

is $2n-1$; therefore the second integral in (7) always converges, at least when $\mathcal{L}\mathbf{u} \in L^q(G)$, where $q > 2n$.

Let now $\mathbf{u}$ be a solution of system (1). Then from (7) we obtain the Cauchy integral for solutions of system (1)

$$\mathbf{u}(M_0) = K\int_S \left[\frac{1}{r^{2n-2}}\,\mathcal{L}\right][\alpha_i \mathcal{P}_i + \beta_i \mathcal{Q}_i]\mathbf{u}\,dS \tag{8}$$

for $M_0 \in G$.

In an analogous way, from (6) we obtain

$$0 = K\int_S \left[\frac{1}{r^{2n-2}}\,\mathcal{L}\right][\alpha_i \mathcal{P}_i + \beta_i \mathcal{Q}_i]\mathbf{u}\,dS$$

for $M_0 \notin \overline{G}$.

Consider now the analogue of the Cauchy-type integral for our system

$$\mathbf{F}(M_0) = K\int_S \left[\frac{1}{r^{2n-2}}\,\mathcal{L}\right][\alpha_i \mathcal{P}_i + \beta_i \mathcal{Q}_i]\mathbf{v}\,dS; \tag{9}$$

here $\mathbf{v}$ is an arbitrary vector prescribed on S, Hölder continuous. It is easy to see that $\mathbf{F}(M_0)$ satisfies our system (1). Analogously to [7,8], one can find formulas for the boundary values of $\mathbf{F}(M_0)$, namely:

$$\mathbf{F}^+(M_1) = \frac{\mathbf{v}(M_1)}{2} + K\int [\,][\,]\,\mathbf{v}\,dS, \tag{10}$$

$$\mathbf{F}^-(M_1) = -\frac{\mathbf{v}(M_1)}{2} + K\int [\,][\,]\,\mathbf{v}\,dS, \tag{11}$$

where the integrals on the right-hand side are understood in the sense of the Cauchy principal value. From these formulas one obtains the necessary and sufficient condition for the vector $\mathbf{v}$ prescribed on S to be the boundary value of some solution of system (1) in the domain G, namely

$$-\frac{\mathbf{v}(M_1)}{2} + K\int [\;][\;]\,\mathbf{v}\,dS = 0. \tag{12}$$

Consider, for our system (1), the boundary-value problem

$$u_i = 0 \quad (i = 2, \dots, 2^n) \quad \text{on } S, \tag{13}$$

where u_i are the components of the vector $\mathbf{u}$. Since each component of the solution is a function harmonic in G, it follows that

$$u_i = 0 \quad (i = 2, \dots, 2^n) \quad \text{in } G. \tag{14}$$

Then the function u_1 must satisfy the equations obtained from our system (1) by putting $u_i = 0$, $i \geq 2$, in it. By calculation one may verify that the resulting system contains only n equations, identically different from zero, which have the form $(\partial/\partial x_\alpha - i\,\partial/\partial y_\alpha)u_1 = 0$. Hence it follows that all solutions of the boundary-value problem (13) for system (1) are described by functions analytic in $\bar{z}_\alpha = x_\alpha - iy_\alpha$.

If now, instead of system (1), we consider the system

$$\mathcal{L}_1\mathbf{u} = \sum_{\alpha=1}^{n}\left(\mathcal{P}_\alpha\frac{\partial}{\partial x_\alpha} - \mathcal{Q}_\alpha\frac{\partial}{\partial y_\alpha}\right)\mathbf{u} = 0, \tag{15}$$

then for it the solutions of the boundary-value problem (13) are functions analytic in the variables $z_\alpha = x_\alpha + iy_\alpha$. For system (15), in a manner analogous to that for system (1), with $\mathcal{Q}_\alpha$ replaced by $-\mathcal{Q}_\alpha$, all formulas valid for system (1) are derived.

Let now $u(M)$ be a function analytic in the variables z_α. Then the vector $\mathbf{F} = \{u(M), 0, \dots, 0\}$ is a solution of system (15); therefore, for it the relations

$$\mathbf{F}(M_0) = \int_S K(M_0, M)\mathbf{F}(M)\,dS \quad \text{for } M_0 \in G, \tag{16}$$

$$0 = \int_S K(M_0, M)\mathbf{F}(M)\,dS \quad \text{for } M_0 \overline{\in} G \tag{17}$$

are valid. Here $K(M_0, M)$ denotes the matrix

$$K(M_0, M) = K\left[\frac{1}{r^{2n-2}}\mathcal{L}\right]\left[\sum(\alpha_k\mathcal{P}_k - \beta_k\mathcal{Q}_k)\right]. \qquad (18)$$

Writing this condition componentwise, we obtain the following analogue of the Cauchy integral formula for functions of several complex variables:

$$u(M_0) = \int_S K_{11}(M_0, M)u(M)\,dS, \qquad (19)$$

$$0 = \int_S K_{j1}(M_0, M)u(M)\,dS, \quad j = 2, \dots, 2^n, \qquad (20)$$

for $M_0 \in G$;

$$0 = \int_S K_{j1}(M_0, M)u(M)\,dS, \qquad (21)$$

$j = 1, \dots, 2^n$, for $M_0 \overline{\in} G$. K_{ji} are the elements of the matrix $K(M_0, M)$.

The formulas for the boundary values of the Cauchy-type integral for solutions of system (15) give the following necessary conditions for the boundary values of an analytic function $u(M_1)$:

$$\frac{u(M_1)}{2} = \int_S K(M_1, M)u(M)\,dS, \qquad (22)$$

$$0 = \int_S K(M_1, M)u(M)\,dS, \qquad (23)$$

$$j = 2, \dots, 2^n.$$

Considering the Cauchy-type integral for system (15) with vector density $\mathbf{F} = (\varphi, 0, \dots, 0)$, it is easy to prove also the sufficiency of conditions (22) and (23).

The result obtained can be formulated as the following assertion.

Theorem. *In order that a function $\varphi(M)$, given on the closed surface S, be the boundary value of some analytic function in the variables $z_\alpha = x_\alpha + iy_\alpha$ in the domain G, it is necessary and sufficient that conditions (22), (23) be satisfied.*

In the case $n = 1$, system (15) splits into two Cauchy–Riemann systems for each complex component, $K_{21} = 0$, and condition (22) reduces to the well-known formula of the theory of functions of one complex variable (7).

Steklov Mathematical Institute
Academy of Sciences of the USSR

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Note: Figure translations are in progress. See original paper for figures.