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Abstract

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MATHEMATICS

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ALGEBRAIC CURVES OVER FUNCTION FIELDS

(Presented by Academician I. M. Vinogradov, April 2, 1968)

Let K be a global field (a number field or a function field of dimension 1); let X be a projective curve, defined over K , nonsingular and geometrically irreducible. With the curve X one can associate the following invariants: the genus g , and the finite set of points of the field K , S , at which the curve X has degenerate reduction. In ⁽¹⁾ I. R. Shafarevich conjectured that the curve X is determined by the collection K, S, g up to a finite number of possibilities, if $g > 1$ and X is a nonconstant curve in the functional case. The aim of the present note is to give a proof of this conjecture for one class of curves over a function field and to establish a connection between the circle of questions considered and the functional analogue of Mordell's conjecture. Our method is customary for Diophantine geometry. First the boundedness of the height of the set of curves under consideration is established, and then, with the aid of a suitable rigidity theorem, its finiteness is proved.

Denote by k the ground field, which we shall assume to be algebraically closed and of characteristic 0. Put $K = k(B)$, where B is a nonsingular projective curve over k of genus g . A curve X of genus $g > 1$, defined over K , may be regarded as a curve of degree $d = 6(2g - 2)$ in projective space of dimension $m = 11g - 12$ (the embedding by means of the sixfold canonical class). Denote by H the Hilbert scheme of curves of degree d in P^m . Each point $x \in H(K)$ determines a curve X_x , defined over K .

Theorem 1. *Let K, S , and $g > 1$ be given. Then there exists a set $\mathcal{E} \subset H(K)$ of bounded height such that for any nonsingular geometrically irreducible K -curve X of genus $g > 1$ and nondegenerate outside S , there is a point $a \in \mathcal{E}$ for which $X_a \simeq X$ over K .*

The main idea of the proof is as follows. By Bézout's theorem, any two curves of degree d in P^m coincide if they intersect in more than d^2 points. Therefore curves are uniquely determined by a sufficiently large set of their points. This reduces the problem to constructing on the curve a K -rational cycle of large degree and to estimating the heights of the points entering into it. Such a cycle is, for example, the general divisor from a suitable multiple canonical class. The

height estimate must be carried out on the minimal fibration associated with the curve X . This fibration is a nonsingular surface V and a flat epimorphism $f : V \rightarrow B$, whose general fiber is isomorphic to X . Minimality means the absence in the fibers V_b of exceptional curves of the first kind. The existence and uniqueness of such a fibration are proved in ⁽²⁾. From the point of view of fibrations, the set S may be defined as the smallest set of points of the curve B outside which the morphism f is smooth.

Lemma 1. *If $q > 1$, then V is a surface of general type (in the sense of ⁽³⁾) and*

$$\Omega_V \cdot \Omega_V \leq 99(gq + 1), \quad s = \text{card}(S).$$

The required height estimate follows easily from the lemma. Let us note, as an additional result, that one can explicitly estimate all numerical invariants of the surface V . In particular, this gives an estimate for the num-

of components of degenerate fibers, their intersection matrices, etc., in terms of g, q , and s . The condition $q > 1$ is necessary for purely technical reasons.

For the formulation of the following result we shall need two definitions. A curve X defined over K will be called nondegenerate if X has nondegenerate reduction at all points of the field K . This means that the morphism $f : V \rightarrow B$ is smooth. In particular, the curves of the form $Y \otimes K$, where Y is a curve defined over k , are nondegenerate. We shall call the curve X nonconstant if it is not of this form over any extension of the field K .

Theorem 2. *The set of nondegenerate nonconstant K -curves of genus $g > 1$ is finite.*

The proof proceeds in two stages. Passing from curves to pencils, we obtain from Theorem 1 the existence of a finite number of families of smooth pencils associated with nondegenerate curves. Each such family is, first of all, a family of algebraic surfaces, which makes it possible to apply the Kodaira-Spencer theory ⁽⁴⁾. Let $f : V \rightarrow B$ be a smooth pencil; T_V the tangent sheaf; $T_{V/B}$ the relative tangent sheaf. If V belongs to the family under consideration, then from the existence of the pencil structure it follows that the Kodaira-Spencer class $c \in H^1(V, T_V)$ belongs to the image of the natural homomorphism $H^1(V, T_{V/B}) \rightarrow H^1(V, T_V)$. To complete the proof it remains to refer to the following lemma:

Lemma 2. *Let Ω_V be the canonical class of the surface V , and Ω_B the canonical class of the curve B . Then:*

1. $T_{V/B} \simeq O_V(-\Omega_V + f^*(\Omega_B))$.
2. *The divisor $\Omega_V - f^*(\Omega_B)$ is ample and, consequently,*

$$H^1(V, T_{V/B}) = 0,$$

if the general fiber of the morphism f is a nonconstant curve and $q > 1$.

From the lemma one can extract an interesting corollary: the self-intersection indices of sections on a smooth nonconstant pencil are negative. This fact is also true for some nonsmooth pencils (for example, if the base B is the projective line). It would be interesting to determine what happens for an arbitrary nonconstant pencil.

Theorem 3. *Suppose that the genus g of the field K is equal to 0 or 1. Then every nondegenerate curve X of genus $g > 1$, defined over K , is constant.*

Proof. If $f : V \rightarrow B$ is the pencil associated with X , then for any automorphism $\sigma : B \rightarrow B$ the pencil $f_\sigma : V_\sigma \rightarrow B$, $V_\sigma = V \times_B B$,

$$f_\sigma = f \times_B \alpha$$

has as its general fiber a nondegenerate curve of genus g . From Theorem 2 it follows that the set of $\sigma \in \text{Aut } B$ such that V_σ is B -isomorphic to V forms in $\text{Aut } B$ a subgroup of finite index. Hence, from the structure of the group $\text{Aut } B$, we obtain that all fibers of the original pencil V are isomorphic. This implies the analytic local triviality of V ⁽⁴⁾. Since the group $\text{Aut } V_b$ is finite for all $b \in B$, the pencil V becomes a direct product over a suitable covering of the base B , and therefore its general fiber is a constant curve.

For fields K of genus 0, this result was obtained by another method by I. R. Shafarevich ⁽¹⁾. Let us also note that arguments analogous to the preceding ones make it possible to obtain from the hypothesis stated at the beginning of the note that a curve of genus $g > 1$ over a field of genus 0 has at least 3 points of degeneration.

We now turn to the consideration of rational points.

Lemma 3. *Let X be a curve of genus $g > 0$, defined over the field $K = k(B)$; let $P \in X(K)$ be a rational point; and let S be the set of points of degeneration over K . Then there exists a covering $\varphi : X' \rightarrow X$, defined over a finite extension K' , unramified outside S , and the following conditions are satisfied:*

1. The degrees of φ and $[K' : K]$ depend only on g .
2. φ is ramified at the point P and unramified outside P .
3. The curve X' is nonsingular outside the set $S \otimes K'$.

The curve X' is constructed as follows. Let $\xi : Y \rightarrow X$ be the covering obtained by multiplication by 2 in the Jacobian of the curve X ⁽⁵⁾. The cycle $\xi^*(P)$ splits into rational points over the field K' . Put $\xi^*(P) = D + P_0$ and turn the generalized Jacobian variety $J_D^{(1)}(Y)$, with the aid of the point P_0 , into a commutative algebraic group. Multiplication by 2 in the generalized Jacobian induces ⁽⁵⁾ a covering $\eta : X' \rightarrow Y$ of the curve Y . It is not difficult to verify that the composition $\varphi = \xi \circ \eta$ and the field K' satisfy all the conditions of the lemma.

From Lemmas 1 and 3 one can obtain

Theorem 4 (Manin ⁽⁶⁾). *The set $X(K')$ of rational points of a nonconstant K -curve X of genus $g > 1$ is finite for any finite extension $K' \supset K$.*

The proof of Theorem 4 on the basis of Lemmas 1 and 3 is completely effective and leads to an estimate for the canonical height

$$h(P) \leq C \exp(\exp 4g),$$

where $C = C(s, q)$ depends only on s and q and is written down explicitly. Using Grauert's proof ⁽⁷⁾ of Theorem 4, one can obtain a sharper estimate

$$h(P) \leq c'(s, q)g^{13}.$$

We shall now show that Theorem 4 follows trivially from the above-mentioned conjecture of I. R. Shafarevich and Lemma 3. In view of Theorem 2 this gives yet another proof for nonsingular curves. If $P \in X(K)$, denote by X_P, K_P the covering constructed in Lemma 3 and its field of definition. If X is a nonconstant curve, then it is easy to verify that all the curves X_P are nonconstant. From the structure of the fundamental group of the curve B it follows that there are only finitely many fields K_P . Therefore, from conditions 1-3 of Lemma 3 and Shafarevich's hypothesis it follows that the set of curves X_P , considered up to isomorphism, is finite. Since for a given curve there exist only finitely many morphisms to a curve of genus greater than 1 (this is the only place where we have used the fact that $g > 1$), we obtain that the number of coverings $X_P \rightarrow X$ is finite, which implies the finiteness of the set $X(K)$ in view of condition 2 of Lemma 3.

As another application of Lemma 3 we give a construction of examples of nonsingular nonconstant curves over a suitable field K . Such examples were first constructed by Kodaira. If Y is a nonsingular curve over a field k and B is its covering, then the curve $Y \otimes K, K = k(B)$, has a nontrivial rational point P . Applying Lemma 3 in this situation, we obtain a nonsingular curve which is nonconstant.

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