ON OPTIMAL SUMMATION OF FOURIER SERIES WITH APPROXIMATE COEFFICIENTS

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Abstract

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ON OPTIMAL SUMMATION OF FOURIER SERIES WITH APPROXIMATE COEFFICIENTS

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In paper (1), following the regularization method developed in (2), a solution is given to the problem of finding an approximate value at a point $x_0 \in [a, b]$ of a function $f(x)$, defined on the interval $[a, b]$, from the approximate values (in the l_2 metric) of its Fourier coefficients $\{c_n\}$ with respect to the orthonormal system of eigenfunctions $\{u_n(x)\}$ of the boundary-value problem:

$$u'' - q^2(x)u + \lambda u = 0, \qquad u(a) = u(b) = 0, \qquad 0 \le q^2(x) \le M. \qquad (*)$$

Let a_n be the exact values of the Fourier coefficients of the function $f(x)$, and let γ_n be their errors such that

$$\sum_{n=1}^{\infty} \gamma_n^2 \le \delta^2.$$

Then $c_n = a_n + \gamma_n$. As the approximate value of the function $f(x)$, in (1) one takes the sum of the series

$$f_\alpha(x) = \sum_{n=1}^{\infty} r(n, \alpha) c_{nu} n(x)$$

with regularizing factors $r(n, \alpha) = 1/(1 + \alpha\lambda_n)$, where λ_n are the eigenvalues of the boundary-value problem $(*)$, and α is the regularization parameter $(\alpha > 0)$.

A. N. Tikhonov posed the problem of finding an optimal, in some sense, summation method, i.e. of choosing the regularizing factors $r(n, \alpha)$. In the present article, a solution of this problem is given in various formulations close to Wiener' s (3) and to the formulations contained in (4–8).

Let, in a finite closed domain $\overline{D}$ of n-dimensional Euclidean space R_n, there be given a complete orthonormal (with weight $\rho(x) > 0$) system of functions

$\{u_n(x)\}$, $x = (x_1, x_2, \dots, x_n)$, and a continuous function $\hat{f}(x)$ in $\overline{D}$, representable by its Fourier series with respect to the system $\{u_n(x)\}$,

$$\hat{f}(x) = \sum_{n=1}^{\infty} a_{nu} n(x), \qquad \text{where} \quad a_n = \int_D \rho(x)\hat{f}(x)u_n(x)\,dx.$$

Suppose that we know approximate values of the Fourier coefficients of this function $\hat{f}(x)$, $c_n = a_n + \gamma_n$, with small (in l_2) errors γ_n,

$$\sum_{n=1}^{\infty} \gamma_n^2 \leq \delta^2.$$

It is required to find an approximate value of the function $\hat{f}(x)$ from the approximate values of its Fourier coefficients $\{c_n\}$. We shall solve this problem by the regularization method.

Consider the functional

$$M^m = \sum_{n=1}^{\infty} (f_n - c_n)^2 + \Omega[f_n],$$

where

$$f_n = \int_D \rho(x)f(x)u_n(x)\,dx, \qquad \Omega[f_n] = \sum_{n=1}^{\infty} f_n^2 m_n,$$

$\{m_n\}$ is a sequence of positive numbers whose order of growth as $n \to \infty$ is not less than

$n^{1+\varepsilon}$, where $\varepsilon > 0$. We shall denote by $\mathfrak{M}$ the class of all such sequences corresponding to different values of ε. The functional $\Omega[f]$ will be called stabilizing. If, as the system $\{u_n(x)\}$, one takes the eigenfunctions of the boundary-value problem $(*)$ and sets $m_n = \alpha\lambda_n$, where λ_n are the eigenvalues of problem $(*)$, then one obtains the stabilizing functional used in paper ([1]).

The function $\tilde{f}(x)$ realizing the minimum of the functional M^m in the class of functions continuous in $\overline{D}$ (the class C_D) will be taken as the approximate value of the function $\hat{f}$. It is easy to see that the Fourier coefficients of the function $\tilde{f}(x)$ are equal to

$$\tilde{f}_n = \frac{c_n}{1 + m_n}, \quad \text{i.e. } \tilde{f}(x) = \sum_{n=1}^{\infty} c_n r(n) u_n(x), \quad \text{where } r(n) = \frac{1}{1 + m_n}.$$

Thus, the summation method is determined by the choice of the sequence $\{m_n\}$. This summation method is stable in the sense that small (in l_2) changes of the coefficients correspond to a small change of $\tilde{f}(x)$ (in the metric of C_D).

To prove this, it is enough first of all to note that the set of functions $f \in C_D$ for which $\Omega[f] \leq d$ is compact in C_D for any $d \geq 0$. Define an operator A acting from C_D into l_2: to a function $f \in C_D$ we assign the sequence of its Fourier coefficients with respect to the system $\{u_n(x)\}$ with weight $\rho(x)$. This mapping is continuous and one-to-one. Consequently, the inverse mapping is also continuous. This proves the assertion on the stability of the summation method.

The errors in the Fourier coefficients, i.e. γ_n, are random numbers, about which we shall assume:

1) $\{\gamma_n\}$ is a sequence of uncorrelated random numbers.

2) The mathematical expectations $E\gamma_n = \overline{\gamma_n} = 0$ for all n. Under these conditions the approximate values of the Fourier coefficients are also random numbers, and $c_n^2 = a_n^2 + \gamma_n^2$. The variances of the random variables γ_n and c_n are the same and are equal to $\sigma_n^2 = \overline{\gamma_n^2}$. The function $\tilde{f}(x)$, realizing the minimum of the functional M^m for a fixed sequence $\{m_n\}$, is a random function.

Let $(\Delta f)_m = \tilde{f}(x) - \hat{f}(x)$, where $\hat{f}(x) = \sum_{n=1}^{\infty} a_{nu} n(x)$. As a measure of the deviation of $\tilde{f}(x)$ from $\hat{f}(x)$ one may take: a) $\overline{(\Delta f)_m^2}$ or b)

$$\int_D \rho(x)\overline{(\Delta f)_m^2}\, dx.$$

Putting $m_n = \alpha\xi_n$ $(n = 1, 2, ...)$, where $\alpha > 0$, $\xi_n > 0$, we obtain

$$\tilde{f}(x) = f_\alpha(x) = \sum_{n=1}^{\infty} \frac{c_n}{1 + \alpha\xi_n} u_n(x).$$

In this case

$$\Omega[f] = \alpha\Omega_1[f] = \alpha \sum_{n=1}^{\infty} f_n^2 \xi_n$$

and $(\Delta f)_m = \Delta f_\alpha$; α is the regularization parameter.

The following formulations of problems are natural:

1_{A}. For a fixed sequence $\{\xi_n\}$, find such a value α_0 of the regularization parameter α for which

$$\overline{(\Delta f_{\alpha_0})^2} = \min_{\alpha} \overline{(\Delta f_\alpha)^2}$$

at a fixed point x_0.

1_{B}. For a fixed sequence $\{\xi_n\}$, find such a value $\alpha = \alpha_0$ for which

$$\int_D \rho(x)\overline{(\Delta f_{\alpha_0})^2}\,dx = \min_{\alpha} \int_D \rho(x)\overline{(\Delta f_\alpha)^2}\,dx.$$

The summation of the Fourier series determined by such a value of α, for a fixed stabilizing functional

$$\Omega_1[f] = \sum_{n=1}^{\infty} f_n^2 \xi_n$$

will be called α-optimal.

IIA. In the class $\mathfrak{M}$ of sequences of positive numbers $\{m_n\}$, find a sequence $\{m'_n\}$ for which $\overline{(\Delta f)^2_{m'}} = \min_{\mathfrak{M}} \overline{(\Delta f)^2_m}$ at the fixed point x_0.

IIB. In the class $\mathfrak{M}$ of sequences $\{m_n\}$ of positive numbers, find a sequence $\{m'_n\}$ for which

$$\int_D \rho(x)\overline{(\Delta f)^2_{m'}}\,dx = \min_{\mathfrak{M}} \int_D \rho(x)\overline{(\Delta f)^2_m}\,dx.$$

The summation of a Fourier series determined by such a sequence $\{m'_n\}$ will be called optimal. This formulation of the problem is analogous to the Wiener problem on optimal filtering ([3]).

Solution of Problems IB and IIB. Since

$$\Delta f_\alpha = \sum_{n=1}^{\infty} \frac{\gamma_n - \alpha a_n \xi_n}{1 + \alpha \xi_n} u_n(x),$$

we have

$$\int_D \rho(x)\overline{(\Delta f_\alpha)^2}\,dx = N(\alpha) = \sum_{n=1}^{\infty} \frac{\sigma_n^2 + \alpha^2 \xi_n^2 \left(\overline{c_n^2} - \sigma_n^2\right)}{(1 + \alpha \xi_n)^2}.$$

From the minimum condition for $N(\alpha)$ we find the solution of problem IB, i.e. the equation for determining α_0:

$$\sum_{n=1}^{\infty} \frac{\sigma_n^2 \xi_n}{(1 + \alpha \xi_n)^3} = \alpha \sum_{n=1}^{\infty} \frac{\xi_n^2 \left(\overline{c_n^2} - \sigma_n^2\right)}{(1 + \alpha \xi_n)^3}.$$

To solve problem IIB one must find the minimum of the function $\varphi(m_1, m_2, \dots, m_n, \dots)$ of the variables $m_1, m_2, \dots, m_n, \dots$, equal to

$$\varphi(m_1, m_2, \dots, m_n, \dots) = \int_D \rho(x)\overline{(\Delta f)_m^2}\, dx = \sum_{n=1}^{\infty} \frac{\sigma_n^2 + \left(\overline{c_n^2} - \sigma_n^2\right) m_n^2}{(1+m_n)^2}.$$

The minimum is attained for $m_n = m_n' = \sigma_n^2 / \left(\overline{c_n^2} - \sigma_n^2\right)$. Consequently, $1/(1+m_n) = 1 - \sigma_n^2/\overline{c_n^2}$. Thus the optimal (in the sense of IIB) summation has the form:

$$\tilde{f}_{\text{op}}(x) = \sum_{n=1}^{\infty} \left(1 - \frac{\sigma_n^2}{\overline{c_n^2}}\right) c_{nu} n(x).$$

To solve problems IA and IIA we shall assume that $\hat{f}(x)$ is a realization of a random process such that:

3) The sequence $\{a_n\}$, determining the function $\hat{f}(x)$, is a sequence of uncorrelated random numbers.

4) The sequences $\{\gamma_n\}$ and $\{a_n\}$ are uncorrelated with each other.

Condition 3) is satisfied, for example, in the case when $\hat{f}(x)$ is a realization of a periodic stationary random process.

Under these additional conditions

$$\overline{(\Delta f_\alpha)^2}\Big|_{x=x_0} = \sum_{n=1}^{\infty} \frac{\sigma_n^2 + \alpha^2 \xi_n^2 \left(\overline{c_n^2} - \sigma_n^2\right)}{(1+\alpha\xi_n)^2} u_n^2(x_0) = \varphi(\alpha, x_0).$$

From the condition that $\varphi(\alpha, x_0)$ be minimal (with respect to α), we find the equation for determining α_0:

$$\sum_{n=1}^{\infty} \frac{\sigma_n^2 \xi_n}{(1+\alpha\xi_n)^3} u_n^2(x_0) = \alpha \sum_{n=1}^{\infty} \frac{\xi_n^2 \left(\overline{c_n^2} - \sigma_n^2\right)}{(1+\alpha\xi_n)^3} u_n^2(x_0).$$

Problem IIA is solved analogously:

$$m_n' = \sigma_n^2 / \left(\overline{c_n^2} - \sigma_n^2\right).$$

Thus, the optimal summation (in the sense of Π_A) has the form

$$\tilde{f}_{\text{op}}(x_0) = \sum_{n=1}^{\infty} \left(1 - \frac{\sigma_n^2}{\overline{c_n^2}}\right) c_{nu} n(x_0).$$

In concrete problems one usually knows only a finite number of possible values of each of the random variables c_n. From these samples we find approximate values of the ratios $\sigma_n^2/\overline{c_n^2}$. Let

$$\left(\frac{\sigma_n^2}{c_n^2}\right)_{\text{pr}} = \frac{\sigma_n^2}{c_n^2}(1+\beta_n), \quad \text{where} \sum_{n=1}^{\infty} \beta_n^2 \leqslant \varepsilon.$$

Since summation with regularizing factors $1/(1+m_n)$ is stable, small deviations in the determination of the numbers σ_n^2/c_n^2 correspond to small deviations of the sum $\tilde{f}(x)_{\text{op}}$. More precisely, if

$$\tilde{f}_{\text{pr}}(x) = \sum_{n=1}^{\infty} \left[1 - \left(\frac{\sigma_n^2}{c_n^2}\right)_{\text{pr}}\right] c_{nu} n(x), \qquad u_n^2(x) \leqslant M \quad (n = 1, 2, \ldots),$$

then

$$\left|\tilde{f}_{\text{op}}(x) - \tilde{f}_{\text{pr}}(x)\right| \leqslant 2M\varepsilon \sum_{n=1}^{\infty} c_n^2.$$

If, as the system of functions $\{u_n(x)\}$, one takes the eigenfunctions of the boundary-value problem

$$\operatorname{div}(k\nabla u) - q^2(x)u + \lambda\rho(x)u = 0, \qquad u|_S = 0 \quad (\text{or } \partial u/\partial n|_S = 0),$$

where S is the boundary of the domain D in which the solution is sought, then the functional $\Omega_1[f]$ may be taken in the form

$$\Omega_1[f] = \int_D \{k(\nabla f)^2 + q^2 f^2\}\, dx$$

or in the equivalent form

$$\Omega_1[f] = \sum_{n=1}^{\infty} f_n^2 \lambda_n,$$

where λ_n are the eigenvalues of the indicated boundary-value problem, and f_n are the Fourier coefficients of the function $f(x)$ with respect to the system $\{u_n(x)\}$. Thus, in this case $m_n = \lambda_n \alpha$. With such a regularizer the problem of summing a Fourier series was considered in the one-dimensional case in ([1]).

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Note: Figure translations are in progress. See original paper for figures.