ON SOLUTIONS OF FINITE FORM OF CERTAIN HILL EQUATIONS

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Abstract

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ON SOLUTIONS OF FINITE FORM OF CERTAIN HILL EQUATIONS

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It is known that the radial wave equation reduces to the form

$$d^2y/d\tau^2 + (\delta + \varepsilon e^{\tau} + \mu e^{2\tau})y = 0 \tag{1}$$

(the Hill equation with imaginary period), for which, under certain relations among δ, ε, μ, solutions in closed form have been found [1].

Solutions of finite form can also be found for the Hill equation with real period, analogous to (1):

$$d^2y/d\tau^2 + (\delta + \varepsilon_c \cos\tau + \varepsilon_s \sin\tau + \mu\cos 2\tau)y = 0. \tag{2}$$

Equation (2), under certain relations among $\delta, \varepsilon_c, \varepsilon_s, \mu$, has solutions in the form of a segment of a Fourier series multiplied by

$$E(\tau) = \exp\left[\int(\lambda_0 + \lambda\sin\tau)d\tau\right]. \tag{3}$$

For $y = E(\tau)f(\tau)$, equation (2) is satisfied if

$$f'' + 2(\lambda_0 + \lambda\sin\tau)f' = (\alpha + 2\beta\cos\tau)f, \tag{4}$$

$$\delta = -(\alpha + \lambda_0^2 + \mu), \qquad \varepsilon_c = -(2\beta + \lambda), \qquad \varepsilon_s = -2\lambda_0\lambda, \qquad \mu = \lambda^2/2.$$

From (4), after the substitution

$$f(\tau) = \sum_{n=0}^{N} \frac{a_n\cos n\tau + b_n\sin n\tau}{c_n\cos(n+\frac{1}{2})\tau + d_n\sin(n+\frac{1}{2})\tau}$$

and elementary transformations, one obtains a system of algebraic equations, the study of which leads to Theorems 1 and 2.

Theorem 1. Let λ_0 be a constant, $\lambda = \sqrt{2\mu}$, $N = 0, 1, \dots$, $L_1 = \operatorname{diag}\{2\lambda_0, 4\lambda_0, \dots, 2N\lambda_0\}$ be a diagonal matrix; B, A be codiagonal matrices:

$$B = \operatorname{codiag}\left\{\begin{array}{cccccc} 1, & \frac{(N+2)\lambda}{2^2}, & \frac{(N+3)\lambda}{3^2}, & \dots & \dots, & \frac{2N\lambda}{N^2} \\ (N-1)\lambda, & (N-2)\lambda, & \dots & \dots, & \lambda & \end{array}\right\},$$

$$A = \operatorname{codiag}\left\{\begin{array}{cccccc} 0, & \frac{(N+1)\lambda}{1}, & \frac{(N+2)\lambda}{2^2}, & \dots & \dots, & \frac{2N\lambda}{N^2} \\ 2N\lambda, & (N-1)\lambda, & (N-2)\lambda, & \dots & \lambda & \end{array}\right\}.$$

If $\varepsilon_c = -(2N+1)\lambda$, $\varepsilon_s = -2\lambda_0\lambda$, and the sum $\delta+\mu+\lambda_0^2$ is equal to an eigenvalue of the matrix

$$A^* = \left[\begin{array}{c|c} A & \begin{matrix} 0 & \dots & 0 \\ & -L_1 & \end{matrix} \\ \hline \begin{matrix} 0 & \\ \vdots & L_1 \\ 0 & \end{matrix} & B \end{array}\right]$$

and to this value there corresponds the eigenvector $(a_0^*, \dots, a_N^*, b_1^*, \dots, b_N^*)$, then equation (2) has the solution

$$y = E(t) \sum_{n=0}^{N} (a_n^* \cos nt + b_n^* \sin nt).$$

Corollary 1.1. If $\varepsilon = -(2N+1)\lambda$, the sum $\delta + \mu$ is equal to an eigenvalue of the matrix A, and to this value there corresponds the eigenvector $(a_0, \dots, a_N)$, then the equation

$$d^2y/d\tau^2 + (\delta + \varepsilon \cos\tau + \mu \cos 2\tau)y = 0 \tag{5}$$

has a 2π-periodic even solution

$$y = e^{-\lambda\cos\tau} \sum_{n=0}^{N} a_n \cos n\tau; \tag{6}$$

if, however, $\delta + \mu$ is equal to an eigenvalue of the matrix B, to which the vector $(b_1, \dots, b_N)$ corresponds, then equation (5) has a 2π-periodic odd solution

$$y = e^{-\lambda\cos\tau}\sum_{n=1}^{N} b_n \sin n\tau. \tag{7}$$

Corollary 1.2. The solutions (6) and (7), as $N \to \infty$, tend to 2π-periodic Mathieu functions of integral order.

Theorem 2. Let $L_2 = \operatorname{diag}\{\lambda_0, 3\lambda_0, \dots, (2N+1)\lambda\}$,

$$C = \operatorname{codiag}\left\{\begin{array}{cccccc} (N+2)\lambda, & (N+3)\lambda, & \dots, & \dots, & \dots, & (2N+1)\lambda \\ (1/2)^2 + (N+1)\lambda, & (3/2)^2, & (5/2)^2, & \dots, & (N-1/2)^2, & (N+1/2)^2 \\ N\lambda, & (N-1)\lambda, & \dots, & \dots, & \lambda & \end{array}\right\};$$

D is the codiagonal matrix obtained from C by changing the sign of λ in the element c_{11}.

If $\varepsilon_c = -2(N+1)\lambda$, $\varepsilon_s = -2\lambda_0\lambda$, and the sum $\delta + \mu + \lambda_0^2$ is equal to an eigenvalue of the matrix

$$C^* = \begin{bmatrix} C & -L_2 \\ L_2 & D \end{bmatrix},$$

to which there corresponds the eigenvector $(c_0^*, \dots, c_N^*, d_0^*, \dots, d_N^*)$, then equation (2) has the solution

$$y = E(\tau)\sum_{n=0}^{N}\left[c_n^* \cos(n+1/2)\tau + d_n^* \sin(n+1/2)\tau\right].$$

Corollary 2.1. If $\varepsilon = -2(N+1)\lambda$, $\delta + \mu$ is equal to an eigenvalue of the matrix C, and to this value there corresponds the eigenvector $(c_0, \dots, c_N)$, then equation (5) has a 4π-periodic even solution

$$y = e^{-\lambda\cos\tau}\sum_{n=0}^{N} c_n \cos(n+1/2)\tau; \tag{8}$$

if, however, $\delta + \mu$ is equal to an eigenvalue of the matrix D, to which there corresponds the eigenvector $(d_0, \dots, d_N)$, then equation (5) has a 4π-periodic odd solution

$$y = e^{-\lambda\cos\tau}\sum_{n=0}^{N} d_n \sin(n+1/2)\tau. \tag{9}$$

Corollary 2.2. The solutions (8) and (9), as $N \to \infty$, tend to 4π-periodic Mathieu functions of integral order.

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Note: Figure translations are in progress. See original paper for figures.

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