ON SPACES OF MAPPINGS INTO A MANIFOLD OF NEGATIVE CURVATURE

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Abstract

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S. I. AL′BER

ON SPACES OF MAPPINGS INTO A MANIFOLD OF NEGATIVE CURVATURE

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The article gives a complete description of spaces of mappings of Riemannian manifolds $V \to \mathfrak{M}^n$ in the case when the Riemannian curvature of $\mathfrak{M}^n$ is negative. In addition to the main type considered in (^{1}), two remaining degenerate cases have been studied.*

1. Harmonic surfaces on Riemannian manifolds. Let $\mathfrak{M}^n, V$ be Riemannian manifolds with metric tensors $g_{ij}(x)$, $g_{\alpha\beta}(v)$, connection coefficients $\Gamma^i_{jk}(x)$, $\hat{\Gamma}^\gamma_{\alpha\beta}(v)$, and curvature tensors $R_{ij,kl}(x)$, $\hat{R}_{\alpha\beta,\gamma\delta}(v)$, respectively. With every smooth mapping $x(v) : V \to \mathfrak{M}^n$ one may associate tensors of mixed type ((^{2}), §115), for example $x^i_\alpha = \partial x^i / \partial v^\alpha$. Denote by ∇_α the total covariant derivative of a mixed tensor. With the aid of the scalar product $(x^i_\alpha, x^j_\beta) = g^{\alpha\beta} g_{ij} x^i_\alpha x^j_\beta$, on each surface $x(v) : V \to \mathfrak{M}^n$ of class $W'_2[V, \mathfrak{M}^n]$ we define the Dirichlet integral

$$D[x(v)] = \frac{1}{2} \oint (x^i_\alpha, x^j_\beta)\, dv,$$

where $dv = {}^*1_V$ is the volume element on V.

Definition 1. A continuous surface $x(v) : V \to \mathfrak{M}^n$ is called **harmonic** if the first variation

$$\delta D[x(v), \eta] = \oint_V (x^i_\alpha, \Delta_\beta \eta^j)\, dv = 0 \tag{1}$$

for any continuous vector field $\eta(x(v))$ of class W'_2, defined on the surface $x(v)$. A harmonic surface $x_0(v)$ is called **stationary** if, in an arbitrarily small neighborhood of it, there exist surfaces for which $D[x(v)] < D[x_0(v)]$.

Theorem 1. *If $x_0(v)$ is a harmonic surface of class $C^2[V, \mathfrak{M}^n]$; $x(v)$ is an arbitrary surface of the same class, and there exists a smooth family of geodesics $z(v, s)$ joining the points $x_0(v)$ and $x(v)$, then*

$$D[x(v)] - D[x_0(v)] = \int_0^1 d\sigma \int_0^\sigma ds \oint_V \left[(\nabla_s z^i_\alpha, \nabla_s z^j_\beta) - \left(R_{lk,m}{}^{i} z^l_s z^m_s z^k_\alpha z^j_\beta\right)\right] dv. \tag{2}$$

Here s is the reduced parameter.

Theorem 2. *If $\mathfrak{M}^n$ is a compact Riemannian manifold of negative curvature, then there are no stationary harmonic surfaces on $\mathfrak{M}^n$.*

2. Descent trajectories. The Euler equation for the variational functional $D[x(v)]$ has the form $\nabla^\alpha \nabla_\alpha x^i = 0$. In order to construct a shortening deformation in the space $C^2[V, \mathfrak{M}^n]$, consider solutions of the parabolic system

$$\partial x^i / \partial t = \nabla^\alpha \nabla_\alpha x^i, \qquad x(v, 0) = \bar{x}(v), \tag{3}$$

corresponding to the Euler equation.

Theorem 3. For a solution $x(v, t)$ of equation (3), the identity

$$\int_{t_0}^{T} dt \oint_V (x_t^i, x_t^j)\, dv = D[x(v, t)] - D[x(v, T)], \tag{4}$$

holds, i.e., the parabolic trajectory is a curve of steepest descent for the Dirichlet functional.

Theorem 4. If $x_1(v, t)$ and $x_2(v, t)$ are two solutions of equation (3), and there exists a smooth family of geodesics $z(v, t; s)$ joining $x_1(v, t)$ and $x_2(v, t)$, then

$$\int_0^1 ds \oint_V (z_s^i, z_s^j)\big|_{t=t_0} dv - \int_0^1 ds \oint_V (z_s^i, z_s^j)\big|_{t=T} dv =$$

$$= \int_{t_0}^{T} dt \int_0^1 ds \oint_V \left[(\nabla_s z_\alpha^i, \nabla_s z_\beta^j) - (R^i_{lk,m} z_s^l z_s^m z_\alpha^k, z_\beta^j) \right] dv. \tag{5}$$

It follows from the theorem that if the curvature of the manifold $\mathfrak{M}^n$ is nonpositive, then there exists a **unique** solution of equation (3) satisfying the prescribed initial condition $x(v, 0) = \bar{x}(v)$.

Suppose that for two solutions at the initial moment $\rho(x_2(v, 0), x_2(v, 0)) < \rho_0$, the elementary length on the manifold $\mathfrak{M}^n$, and therefore the points $x_1(v, 0)$ and $x_2(v, 0)$ can be joined by a unique shortest geodesic $z(v, 0; s)$. The field of geodesics $z(v, 0; s)$ can be continued to a smooth family of geodesics $z(v, t; s)$ joining $x_1(v, t)$ and $x_2(v, t)$. Then from (5) we obtain

Corollary. If the curvature of $\mathfrak{M}^n$ is nonpositive, then the function $\Phi(t) =$

$$= \int_0^1 ds \oint_V (z_s^i z_s^j)\, dv$$

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* The results were reported at the International Congress of Mathematicians in Moscow in 1966.

is a monotonically decreasing function of t.

Similarly to (5), we obtain the identity for a solution of equation (3)

$$\oint_V (x^i_t, x^j_t)\big|_{t=t_0}\, dv - \oint_V (x^i_t, x^j_t)\big|_{t=T}\, dv =$$

$$= \int_{t_0}^{T} dt \oint_V \left[(\nabla_t x^i_\alpha, \nabla_t x^j_\beta) - R^i_{lk,m} x^l_t x^m_t x^k_\alpha x^j_\beta\right] dv. \tag{6}$$

Finally, for a solution $x(v,t)$ of equation (3), the identity

$$\frac{1}{2}\left[\nabla^\gamma \nabla_\gamma (x^i_\alpha, x^j_\beta) - \frac{\partial}{\partial t}(x^i_\alpha, x^j_\beta)\right] =$$

$$= (\nabla_\gamma x^i_\alpha, \nabla_\theta x^j_\beta) - (R^i_{lk,m} x^l_\theta x^m_j x^k_\alpha, x^j_\beta) - g_{ij} \hat{R}^{\alpha\beta} x^i_\alpha x^j_\beta, \tag{7}$$

holds, which is used for an a priori estimate of the derivatives of the solution.

Remark. The scalar product of mixed tensors is defined as the complete contraction of the mixed tensors with the metric tensors of the manifolds $\mathfrak{M}^n$ and V.

Let $\mathfrak{M}^n$ be a manifold of nonpositive curvature. From (7) follows the inequality

$$\frac{\partial}{\partial t}(x^i_\alpha, x^j_\beta) - \nabla^\gamma \nabla_\gamma (x^i_\alpha, x^j_\beta) \le c\,(x^i_\alpha, x^j_\beta). \tag{8}$$

Apply the maximum principle to the solution $x(v,t)$ in the cylinder $V \times [0,T]$. Then we obtain the a priori estimate

$$(x^i_\alpha, x^j_\beta) \le e^{aT} \max_{v\in V,\ t=0} (x^i_\alpha, x^j_\beta). \tag{9}$$

Suppose that $\mathfrak{M}^n$ is a compact manifold. Embed $\mathfrak{M}^n$ by a sufficiently smooth mapping into the Euclidean space R^N and construct in

the normal bundle $\mathfrak{N}$ (^{3,4}) of the manifold $\mathfrak{M}^n$. Let $\mathfrak{N}_r \subset \mathfrak{N}$ be a subbundle of balls of radius r. If r is sufficiently small, then $\mathfrak{N}_r$ forms a tubular neighborhood of $\mathfrak{M}^n$ in R^N. Introduce in $\mathfrak{N}_r$ a Riemannian metric, taking the metric in the fibers to be Euclidean and in the base to coincide with the metric in $\mathfrak{M}^n$.

If the curvature of $\mathfrak{M}^n$ is nonpositive, then the curvature of the manifold $\mathfrak{N}_r$ is also nonpositive. Therefore, for the Dirichlet functional and for the parabolic system in the space of mappings $V \to \mathfrak{N}_r$, the previously proved theorems and estimates are valid.

Consider some local coordinate system (x, y) of the fibering $\mathfrak{N}_r$. Here x are local coordinates in a neighborhood on $\mathfrak{M}^n$, and y are coordinates in the fiber R^{N-n}. In these coordinates the system of Euler equations for the Dirichlet functional splits:

$$\nabla^{\nu}\nabla_{\nu}x^i = 0, \qquad \Delta y^j = 0, \qquad i = 1, \dots, n, \quad j = 1, \dots, N-n. \tag{10}$$

Analogously, the corresponding system of parabolic equations has the form

$$\partial x^i/\partial t = \nabla^{\nu}\nabla_{\nu}x^i, \qquad \partial y^j/\partial t = \Delta y^j, \tag{11}$$

whence it follows that if at the initial moment $y^j(v, 0) \equiv 0$, then for all $t \geqslant 0$ one has $y^j(v, t) \equiv 0$. Therefore, in order to study the solutions of system (3), it suffices to study the solution of the parabolic system (11).

Introduce in $\mathfrak{N}_r$ coordinates $z_1, \dots, z_N$ of the space $R^N \supset \mathfrak{N}_r$ and, using the fundamental solution (^{5}) of the operator $\partial/\partial t - \overline{\nabla}^{\nu}\overline{\nabla}_{\nu}$ on V, replace the parabolic system by a system of nonlinear integral equations. Applying the a priori estimates found, we obtain (^{1,6}):

Theorem 5. Let $\mathfrak{M}^n$ be a compact Riemannian manifold of nonpositive curvature, and let the surface $\bar{x}(v) : V \to \mathfrak{M}^n$ belong to $C^2[V, \mathfrak{M}^n]$. There exists, moreover a unique, solution of the parabolic system (11) of class $C^{2,1}[V \times [0, \infty), \mathfrak{M}^n]$ with initial function $\bar{x}(v)$.

Theorem 6. For each solution $x(v, t) \in C^{2,1}[V \times [0, \infty), \mathfrak{M}^n]$ there exists a unique harmonic surface $x_0(v)$ for which

$$\lim_{t \to \infty} \|x(v, t), x_0(v)\|_{C^1[V, \mathfrak{M}^n]} = 0.$$

3. Topology of the space of mappings

Applying the results of the paper (^{1}), we obtain the following theorem.

Theorem 7. Let $\mathfrak{M}^n$ be a compact Riemannian manifold of negative curvature. Then for an arbitrary compact Riemannian manifold V, any connected component K of the space of mappings $C^2[V, \mathfrak{M}^n]$:

1) either has the homotopy type of the manifold $\mathfrak{M}^n$, and every harmonic surface $x_0(v)$ from K maps V to a single point of the manifold $\mathfrak{M}^n$;
2) or has the homotopy type of a circle, and all harmonic surfaces $x_0(v)$ from K map V, with equal value of the Dirichlet integral, into one and the same closed geodesic of the manifold $\mathfrak{M}^n$;
3) or has the homotopy type of a point, and the component K contains a unique harmonic surface $x_0(V)$.

Proof. Suppose that the functional $D[x(v)]$ has in K two points of minimum $x_0(v)$ and $x_1(v)$. Analogously to the proof of the barrier theorem (^{1}), we show that in this case either the critical value of the homotopy class of curves joining $x_0(v)$ to $x_1(v)$,

$$q > \max(D[x_0(v)], D[x_1(v)]), \tag{12}$$

and there exists a stationary harmonic surface, or $q = D_{[x_0(v)]} = D_{[x_1(v)]}$ and there exists a continuous family of harmonic surfaces joining $x_0(v)$ and $x_1(v)$. But from Theorem 2 it follows that the first case is impossible. In the second case, according to Theorem 1, either all harmonic-

surfaces from K map V into points of the manifold $\mathfrak{M}^n$, or map V onto one and the same closed geodesic.

Remark. From identity (7) it follows that if the Ricci curvature of the manifold V is positive, then the space $C^2[V, \mathfrak{M}^n]$ consists of a single component of type 1. If the Ricci curvature of the manifold V is nonnegative, then the components K can only be of type 1 and type 2.

Corollary. *If two elements of the fundamental group of a manifold of negative curvature $\mathfrak{M}^n$ commute with each other, then they belong to one cyclic subgroup.*

Scientific Research Institute
of Applied Mathematics and Cybernetics
at Gorky State University
named after N. I. Lobachevsky

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Note: Figure translations are in progress. See original paper for figures.