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Abstract

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MATHEMATICS

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ASYMPTOTIC SOLUTION OF SOME SYSTEMS OF DIFFERENTIAL EQUATIONS WITH DEVIATING ARGUMENT

(Presented by Academician A. N. Tikhonov on 14 III 1967)

Statement of the problem. In papers ([1,2]) the possibility was studied of applying the averaging method to systems with deviating argument of the form

$$\dot{x}(t) = \varepsilon X[x(t), x(t-\Delta), t], \tag{1}$$

where $x(t)$ is an n-dimensional vector function; $\varepsilon > 0$ is a small parameter; Δ (the delay) is a fixed positive quantity. Systems of the form (1) were studied in the first approximation (with respect to the parameter ε). It was shown that, on an interval of order $1/\varepsilon$, the estimate

$$|x - \xi| = o(1), \tag{2}$$

holds, where x is a solution of system (1), and ξ is a solution of the system $\dot{\xi} = \varepsilon \overline{X}(\xi)$, which in what follows we shall call the **averaged system of the first approximation**. Here

$$\overline{X}(\xi) = \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t+T} X(\xi, \xi, t)\, dt. \tag{3}$$

Systems of type (1) without delay are called systems in **standard form** ([3]).

In papers ([4,5]) the averaging method was applied to some more general systems with constant delay.

In the present note we report results concerning the development and justification of a general scheme for constructing approximations of arbitrary order for systems with delayed argument of the form (1) and for some systems of neutral type. The delay Δ may be either constant or variable.

Main results.

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I. **Systems with delayed argument.**
In the general case system (1) is considered in the form

$$\dot{x}(t) = \varepsilon X[x(t), x(t-\Delta), t, \varepsilon] \qquad (t \geq t_0). \tag{4}$$

1. **Constant delay** ($\Delta \equiv \text{const}$). The initial data for system (4) are prescribed on the interval $[t_0 - \Delta, t_0]$.

To construct higher approximations we seek a change of variables of the form

$$x = \xi + \varepsilon u_1(\xi, t) + \varepsilon^2 u_2(\xi, t) + \cdots + \varepsilon^n u_n(\xi, t) + \cdots, \tag{5}$$

where ξ is a solution of the averaged system

$$\dot{\xi} = \varepsilon \overline{X}_1(\xi) + \varepsilon^2 A_2(\xi) + \cdots + \varepsilon_n^{nA}(\xi) + \cdots. \tag{6}$$

Here $X_1 \equiv X|_{\varepsilon=0}$.

Restricting ourselves in the right-hand sides of (6) to terms of order ε^n, we obtain the so-called averaged system of the n-th approximation. Its solution is ...

is the n-th approximation for the solutions of system (6). Substituting it into (5) and discarding terms of order ε^n and higher, we obtain the n-th approximation for the solutions of system (4).

The principal requirements of the theorem on the second approximation are the following.

Uniformly with respect to ξ and t_0, in a certain domain there exist limits independent of t_0—mean values of the form (3)—for a number of known functions of the right-hand sides of system (4).

On the interval $[t_0, L/\varepsilon]$, where L is an arbitrarily large positive number, there exists a solution of system (4) and a unique solution of the averaged system of the second approximation.

If these, as well as certain other conditions concerning the smoothness of the right-hand sides of (4), are satisfied, then uniformly on the interval $[2\Delta + t_0, L/\varepsilon]$ the estimates

$$|x - \xi - \varepsilon u_1(\xi, t)| = o(\varepsilon). \tag{7}$$

hold. Here ξ is the solution of the averaged system of the second approximation with initial data at the point $2\Delta + t_0$, obtained by approximate integration of system (4) on the interval $[t_0, 2\Delta + t_0]$ with accuracy up to quantities of order ε inclusive, and

$$u_1(\xi, t) = \int_{t_0+2\Delta}^{t} [X_1(\xi, \xi, t) - \overline{X}_1(\xi)]\, dt.$$

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Theorems on higher approximations are formulated in analogous terms.

2. Variable delay $(\Delta \equiv \Delta(t))$**.** The initial data for system (4) are prescribed on the initial set consisting of the point t_0 and those values $t - \Delta(t)$ for which $t - \Delta(t) < t_0$ when $t \geq t_0$; $\Delta(t)$ is a continuous nonnegative function bounded for all t by a fixed number $\Delta_0 > 0$. The averaged system of the first (second) approximation is considered on the interval $[\gamma^*(t_0), L/\varepsilon]$ $([\gamma^*(\gamma^*(t_0)), L/\varepsilon])$. Here $\gamma^*(z)$ is a function of the point z, determining the moment starting from which the influence of the states of the system preceding the moment z ceases. The point $\gamma^*(\gamma^*(t_0))$ plays the same role with respect to the moment $\gamma^*(t_0)$. For more detail on the properties of the function $\gamma^*(z)$, see, for example, ([6]).

The initial data for the averaged system of the first (second) approximation are prescribed at the point $\gamma^*(t_0)$ $(\gamma^*(\gamma^*(t_0)))$ and are determined by approximate integration of system (4) on the interval $[t_0, \gamma^*(t_0)]$ $([t_0, \gamma^*(\gamma^*(t_0))])$ with accuracy up to terms of order ε^0 (ε^1) inclusive. In this case, on the intervals $[\gamma^*(t_0), L/\varepsilon]$ and $[\gamma^*(\gamma^*(t_0)), L/\varepsilon]$, the estimates (2) and (7), respectively, hold. Higher approximations are considered analogously.

3. Delay depending on the unknown function $(\Delta \equiv \Delta(t, x(t)))$**.** The function $\Delta(t, x(t))$ is assumed to be uniformly bounded $(0 \leq \Delta(t, x(t)) \leq \Delta_0)$. The initial data of system (4) are prescribed on the initial set $[t_0 - \Delta_0; t_0]$. Under certain smoothness conditions on the function $\Delta(t, x)$ (for the theorem on the second approximation, for example, the existence of bounded partial derivatives up to the second order inclusive is required), the estimates (2) and (7) will hold on the intervals $[t_0 + \Delta_0, L/\varepsilon]$ and $[t_0 + 2\Delta_0, L/\varepsilon]$, respectively. Higher approximations are considered analogously.

4. Delays depending on the parameter ε**.** If the dependence on the parameter ε does not violate the uniform boundedness of the functions Δ considered in the preceding points, then all the results remain valid.

We now consider system (4) under the following conditions. The nonnegative function $\Delta(\varepsilon)$ may increase without bound as $\varepsilon \to 0$, but in such a way that

$$\lim_{\varepsilon \to 0} \varepsilon \Delta(\varepsilon) = 0.$$

In this case, the change of variable

$t = \Delta(\varepsilon) \cdot \theta$ brings (4) to the form

$$dx/d\theta = \mu X[x, x(\theta - 1), \theta, \mu], \tag{8}$$

where $\mu = \varepsilon \Delta(\varepsilon)$ is a new small parameter. For this system the results of Sec. 1 are valid. In the second approximation, for example, for $t \in [2\Delta(\varepsilon) + t_0,\ L/\varepsilon]$ the estimates

$$|x - \xi - \mu u_1(\theta, \xi)| = o(\mu)$$

hold.

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In the cases $\Delta \equiv \Delta(t,\varepsilon)$ and $\Delta \equiv \Delta(t,x,\varepsilon)$, results similar to those of Secs. 2, 3 are obtained under the assumption of the existence of such a positive function $\Delta_1(\varepsilon)$ that the ratio $\Delta/\Delta_1(\varepsilon)$ is uniformly bounded by a constant independent of ε and

$$\lim_{\varepsilon\to 0} \varepsilon\Delta_1(\varepsilon) = 0.$$

II. **Systems of neutral type.** The following systems of neutral type were considered:

$$\dot{x}(t) = \varepsilon X[x(t), x(t-\Delta), x'(t-\Delta), t, \varepsilon]. \tag{9}$$

Here $x'(t-\Delta)$ means the derivative of the function $x(z)$ taken at the point $z = t-\Delta$. For delays $\Delta \equiv \text{const}$, $\Delta \equiv \Delta(t)$, $\Delta \equiv \Delta(t,x)$, results analogous to those of Secs. 1-3 have been obtained.

The asymptotic solutions of systems (9) have a somewhat different structure when the delay depends on the parameter ε and increases without bound as $\varepsilon \to 0$.

Let us consider the simplest case of system (9) with a delay depending only on ε ($\Delta \equiv \Delta(\varepsilon)$). Suppose again that

$$\lim_{\varepsilon\to 0} \varepsilon\Delta(\varepsilon) = 0.$$

The change of variable $t = \Delta(\varepsilon)\cdot\theta$ brings (9) to a system containing two small parameters:

$$\frac{dx}{d\theta} = \mu X\left[x(\theta), x(\theta-1), \nu\frac{dx}{d\theta}(\theta-1), \theta, \mu, \nu\right], \tag{10}$$

where $\mu = \varepsilon\Delta(\varepsilon)$, $\nu = 1/\Delta(\varepsilon)$.

The formal change of variables used for constructing an asymptotic solution for such a system has the form

$$x = \xi + \mu u_{10} + \mu^2 \sum_{i,j=0}^{\infty} \mu^i \nu^j u_{i+2,j}(\xi,\theta).$$

Since the relation between the parameters μ and ν is known, it is advisable to compose the asymptotic formula for a particular approximation with respect to the parameter ε by selecting terms according to their order of smallness with respect to ε.

Solutions of systems (9) with delay $\Delta \equiv \Delta(t,\varepsilon)$ or $\Delta \equiv \Delta(t,x,\varepsilon)$ are constructed according to the same scheme, provided the condition of uniform boundedness in t and x indicated in Sec. 4 is satisfied.

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Note: Figure translations are in progress. See original paper for figures.

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