ON THE RELATIONSHIP BETWEEN CERTAIN QUALITY INDICES IN LINEAR STATIONARY CONTROLLED SYSTEMS

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Abstract

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ON THE RELATIONSHIP BETWEEN CERTAIN QUALITY INDICES IN LINEAR STATIONARY CONTROLLED SYSTEMS

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A linear stationary controlled system is considered, described by the equation

$$N(y) \equiv c_0 y^{(n)} + c_1 y^{(n-1)} + \dots + c_{n-1} y' + y = x \tag{1}$$

with initial conditions

$$y(0) = \dots = y^{(n-1)}(0) = 0. \tag{2}$$

The input action $x(t)$, which the system must reproduce, is not known in advance and belongs to the class F of functions with bounded rate of change $x(0) = 0; \quad x(t) = v(t); \quad |v(t)| \le 1$.

Let $y_x(t)$ be the solution of equation (1) with initial conditions (2); $\varepsilon_x(t) = x(t) - y_x(t)$ is the error in reproducing the input signal. One of the quality indices of the system is the greatest accumulated error ([1,2])

$$\varepsilon_{\max}(T) = \max_{x \in F} |\varepsilon_x(T)|; \qquad \varepsilon_\infty = \lim_{T \to \infty} \varepsilon_{\max}(T).$$

In the present paper the relation of $\varepsilon_{\max}(T)$ and ε_∞ to such quality indices of the system as the degree of stability and oscillatory character is investigated ([3–6]).

It is assumed that all roots $p_k = -\alpha_k + i\beta_k$ of the characteristic equation $N(p) = 0$ lie in the left half-plane. The degree of stability and, respectively, the oscillatory character of the system are the quantities

$$a = \min \alpha_j; \qquad \mu = \max |\mu_j|; \qquad \mu_j = \beta_j / \alpha_j; \qquad j = 1, \dots, n.$$

For a first-order equation $c_0 y' + y = x$, the degree of stability is $a = c_0^{-1}$ and

$$\varepsilon_{\max}(t) = a^{-1}[1 - \exp(-at)]; \qquad \varepsilon_\infty = a^{-1}.$$

For the second-order equation $c_0 y'' + c_1 y' + y = x$ with degree of stability a and oscillatory character μ,

$$\varepsilon_\infty = 2a^{-1}\{(1+\mu^2)^{-1} + (1+\mu^2)^{-1/2} \times \exp(-\mu^{-1}(\pi - \psi))[1 - \exp(-\pi\mu^{-1})]^{-1}\}; \qquad \psi = \operatorname{arctg}\mu; \qquad \varepsilon_\infty \leq 2a^{-1};$$

$$\varepsilon_\infty = 2a^{-1} \text{ for } \mu = 0; \qquad \lim_{\mu\to\infty} \varepsilon_\infty = 2a^{-1}\pi^{-1}.$$

In this case, as $\mu \to \infty$, the quantity ε_∞ remains bounded. It is possible, however, to indicate systems for which, for any fixed degree of stability, $\varepsilon_\infty \to \infty$ if $\mu \to \infty$. For example, a system described by a fourth-order equation with two pairs of multiple roots $p_{1,2} = -\alpha + i\beta$; $p_{3,4} = -\alpha - i\beta$. Such an equation, for the case $x = 0$ and nonzero initial conditions, was considered in ([7]).

An upper estimate of the quantities $\varepsilon_{\max}(t)$ and ε_∞ as functions of a and μ for $n \geq 3$ is given by the theorem:

Theorem 1. *If the degree of stability of the closed system (1) is not less than a and the oscillatory character does not exceed μ, then for input actions $x(t)$ from the class F*

$$\varepsilon_{\max}(t) < (2a\gamma^2)^{-1} A_n(\gamma,\mu)[1 - \exp(-a\gamma t)]; \qquad \varepsilon_\infty < (2a\gamma^2)^{-1} A_n(\gamma,\mu).$$

Here γ is an arbitrary parameter satisfying the inequality $0 < \gamma < 1$,

$$A_n(\gamma,\mu) = (1-\gamma)^{2-n} \max\{\tau_1(1-\gamma)^{-1}, \tau_2\}, \quad \text{if } \mu \leq \varphi_1(\gamma) =$$

$$= (1-\gamma)^{-1}\left(1 + \sqrt{1 - (1-\gamma)^2}\right).$$

If $\mu > \varphi_1(\gamma)$, then

$$A_n(\gamma,\mu) = \left[(1+\mu^2)(2\mu(1-\gamma))^{-1}\right]^{(n-2)/2} \max\{\tau_1(1-\gamma)^{-1}, \tau_2\} \quad \text{for even } n; \tag{3}$$

$$A_n(\gamma,\mu) = \max\left\{\left[(1+\mu^2)(2\mu(1-\gamma))^{-1}\right]^{(n-1)/2}\tau_1,\ \left[(1+\mu^2)(2\mu(1-\gamma))^{-1}\right]^{(n-3)/2}\tau_2(1-\gamma)^{-1}\right\} \quad \text{for odd } n;$$

$$\tau_2 = \frac{1}{\sqrt{2}}(1+\mu^2)\left[(1-\gamma)^2 - \gamma^2 - \mu^2 + \right.$$

$$\left. + \sqrt{(1-2\gamma)^2 + 2\mu^2((1-\gamma)^2 + \gamma^2) + \mu^4}\right]^{-1/2} \tag{4}$$

Fig. 1: block diagram of a closed-loop system

Figure 1: Fig. 1: block diagram of a closed-loop system

for $\gamma < (1+\sqrt{2})^{-1}$ or $\gamma \geq (1+\sqrt{2})^{-1}$ and $\mu > \varphi_2(\gamma)$;

$$\tau_2 = \gamma(1+\mu^2)\left((1-\gamma^2)+\mu^2\right)^{-1}, \quad \text{if } \gamma \geq (1+\sqrt{2})^{-1} \text{ and } \mu \leq \varphi_2(\gamma);$$

$$\varphi_2(\gamma) = \sqrt{-(1-\gamma)^2 - \gamma^2 + \gamma\sqrt{4(1-\gamma)^2+\gamma^2}};$$

$$\tau_1 = 1 \text{ for } \gamma \leq \tfrac{1}{2}; \qquad \tau_1 = \gamma(1-\gamma)^{-1} \text{ for } \gamma > \tfrac{1}{2}. \tag{5}$$

Putting $x(t) = t$, we obtain a lower estimate for ε_∞.

Theorem 2. Let the degree of stability of system (1) be a, and let the oscillation not exceed μ. Then, for prescribed inputs from the class F,

$$\varepsilon_\alpha > a^{-1}\min\{1,\, 2(1+\mu^2)^{-1}\}. \tag{6}$$

In the case when $|x(t)| \leq 1$, the lower estimate

$$y_\infty = \max_x |y_x(\infty)|$$

is given in (8). As the example below shows, in inequality (6) one cannot dispense with the requirement that μ be bounded.

Example. The controlled system is described by the equation

$$c_0 y''' + c_1 y'' + c_2 y' + y = x,$$

whose characteristic equation has the roots

$$p_1 = -\alpha_1, \qquad p_{2,3} = -\alpha \pm i\beta; \qquad \alpha_1 > 2\alpha > 0;$$

$$\beta = \tau_3\alpha_1.$$

Then

$$\varepsilon_\infty \leq \alpha_1^{-1} + (\tau_3\alpha)^{-1}.$$

Fig. 1

For any fixed, however small, degree of stability α, the quantities $\varepsilon_{\max}(t)$ and ε_∞ can be made arbitrarily small by choosing α_1 and τ_3 sufficiently large.

Let us consider the closed astatic system, widespread in engineering, shown in Fig. 1. Here

$$W(p) = K/pL(p)$$

is the transfer function of the open-loop system and $L(0) = 1$. It is assumed that all roots

$$p_j = -\alpha_j(1 + i\mu_j)$$

of the polynomial $L(p)$ of order $n-1$ are known. Such an assumption is admissible, since often the controlled system contains a number of series-connected elements, each of which is described by an equation of low order—first or second. Under these conditions we shall give estimates of the degree of stability and oscillation of the closed astatic system.

Theorem 3. If the degree of stability of the polynomial $L(p)$ is equal to a_0, then, for

$$a|L(-a)| \leq K \leq \lambda_1 a|L(-a)| \tag{7}$$

in the closed system the degree of stability is not less than $a = \sigma a_0$. Here

$$\sigma = [R_1(n-1) + n]^{-1}; \qquad R_1 = \left\{ \lambda_1^2 \prod_{j=1}^{n-1}{}' \left(\frac{1+\nu_j^2}{2\nu_j} \right) - 1 \right\}^{1/2};$$

$$\nu_j = |\mu_j| \left(1 - \frac{a_0}{n\alpha_j} \right)^{-1}; \tag{8}$$

$\lambda_1 \geq 1$ is an arbitrary number; the prime sign means that only those factors for which $\nu_j > 1$ are included in the product; if all $\nu_j \leq 1$, then

$$R_1 = \sqrt{\lambda_1^2 - 1}.$$

Corollary. If $q < n-1$ roots of $L(p)$ lie to the left of the line $\operatorname{Re} p = -a_2$; $a_2 > a_0$, then in the closed-loop system, for $a^*|L(-a^*)| \leq K \leq \lambda_1 a^*|L(-a^*)|$, the degree of stability is not less than a^*. Here a^* is the smallest root, quadratic in a, of the equation

$$(n-1+\tau_4)a^2 - a[(q+\tau_4)a_0 + (n-1-q+\tau_4)a_2] + \tau_4 a_0 a_2 = 0; \quad \tau_4 = (R_1+1)^{-1}.$$

The numbers ν_j in formula (8) for R_1 are determined by the relation

$$\nu_j = |\mu_j| \left(1 - \frac{a_0}{\alpha_j(n-q)} \right)^{-1}.$$

An estimate of the oscillation μ in the closed-loop system is given by the theorem:

Theorem 4. If the value of the gain coefficient K satisfies inequalities (7), then the oscillation of the closed-loop astatic system does not exceed the value

$$\mu = \sqrt{R_2^2(a)(1 + \mu_0^2) - 1}.$$

Here $R_2(a)$ is an arbitrary positive number greater than one satisfying the inequality

$$R\prod_{j=1}^{n-1}\left(1-\frac{1+\mu_j^2}{R^2(1+\mu_0^2)}\right)^{1/2} > \frac{\lambda_1}{\sqrt{1+\mu_0^2}}\prod_{j=1}^{n-1}\sqrt{\left(1-\frac{a}{\alpha_j}\right)^2+\mu_j^2} = G;$$

$$\mu_0 = \max|\mu_j|; \qquad j = 1, \dots, n-1. \tag{9}$$

Corollary. If the conditions of the corollary to Theorem 3 are fulfilled, then the oscillation of the closed-loop system is not greater than

$$\mu = \sqrt{R_2^2(a^*)(1+\mu_0^2)-1}.$$

Since the left-hand side of inequality (9) increases monotonically with increasing R, determining $R_2(a)$ is not difficult. The location of the roots of the closed-loop system as a function of the magnitude of the gain coefficient was investigated by the root-hodograph method, for example, in ([9–11]).

Let us consider the question of lowering the order of differential equations ([12,13]), estimating the greatest possible difference between the solutions of the original and the "shortened" equations. Suppose that the behavior of the system under the influence of a control signal $u(t)$ is described by an equation of order n:

$$L_1(y_u) = u; \qquad y_u(0) = \dots = y_u^{(n-1)}(0) = 0; \qquad |u(t)| \le 1. \tag{10}$$

The characteristic polynomial $L_1(p)$ has the form $L_1(p) = pM(p)Q(p)$. Here $M(p)$ and $Q(p)$ are polynomials of orders m and q; $m+q+1=n$. It is assumed that the roots of $L_1(p)$ are known. The roots p_j ($j = 1, \dots, m$) of the polynomial $M(p)$ and the roots p_j ($j = m+1, \dots, m+q$) of the polynomial $Q(p)$ satisfy the conditions

$$-a_1 < -\alpha_j = \operatorname{Re} p_j < -a_0 < 0 \quad (j = 1, \dots, m);$$
$$-\alpha_j = \operatorname{Re} p_j < -a_2 < -a_1 \quad (j = m+1, \dots, m+q). \tag{11}$$

By $z_u(t)$ is denoted the solution of the "shortened" equation $L_2(z_u) = u$ with zero initial conditions. Here $L_2(p) = pM(p)$. Let

$$\varepsilon_u^{(s)} = y_u^{(s)} - z_u^{(s)}, \qquad \varepsilon_{\max}^{(s)}(t) = \max_{|u|\le 1}|y_u^{(s)}(t) - z_u^{(s)}(t)|, \quad s = 0, \dots, m-1.$$

The theorem estimates the maximum error when the order is lowered.

Theorem 5. For any admissible control $u(t)$ from (10),

$$|\varepsilon_u^{(s)}(t)| \le \varepsilon_{\max}^{(s)}(t) \le D_s(1-e^{-at}); \qquad D_s = \frac{1}{\pi a}B(a)\xi_{s+1}^*\prod_{j=1}^{m}\eta_j\prod_{j=1}^{s}\xi_j.$$

Here $0 < a < a_0/2$ is an arbitrary number;

$$L(a) = \frac{g_0 - g_1}{a} \operatorname{arctg} \frac{\omega_0}{a} + \frac{g_1 \pi}{2a} + \frac{1}{2} g_2 \ln\left(1 + \frac{\omega_0^2}{a^2}\right); \qquad g_0 = \left|\frac{1}{Q(-a)} - 1\right|;$$

$$g_1 = 1 + \prod_{j=m+1}^{m+q} \eta_j; \qquad g_2 = \frac{\sqrt{2}}{2(\sqrt{2}-1)} \prod_{j=m+1}^{m+q} \eta_j \sum_{j=m+1}^{m+q} \frac{1}{\alpha_j - a}; \qquad \omega_0 = \frac{g_1 - g_0}{g_2}. \tag{12}$$

To the root $p_j = -\alpha_j(1 + \mu_j)$ of the polynomial $L_1(p)$ there corresponds the factor

$$\eta_j = \left[\frac{1 + \mu_j^2}{(1 - \gamma_j)^2 + \mu_j^2}\right]^{1/2}, \quad \text{if } \mu_j < 1 - \gamma_j; \qquad \eta_j = \left[\frac{1 + \mu_j^2}{2\mu_j(1 - \gamma_j)}\right]^{1/2},$$

$$\text{if } \mu_j > 1 - \gamma_j;$$

$$\zeta_j = \frac{\alpha_j}{\eta_j} \left[\frac{1 + \mu_j^2}{2\mu_j(1 - \gamma_j)} \sqrt{(\gamma_j^2 + (1 - \gamma_j)^2 + \mu_j^2)^2 - 4\gamma_j^2(1 - \gamma_j^2)}\right]^{1/2},$$

if $\mu_j^2 > (1 - \gamma_j)^2 - \gamma_j^2$;

$$\zeta_j = \frac{\alpha_j}{\eta_j} \sqrt{1 + \mu_j^2}, \quad \text{if } \mu_j^2 \leqslant (1 - \gamma_j)^2 - \gamma_j^2; \qquad \gamma_j = \frac{a}{\alpha_j}.$$

The roots of the polynomial $M(p)$ have been renumbered in increasing order of the quantities ζ_j. If $s + 1 < m$, among the first s roots of $M(p)$ there is an even number of complex roots, and p_{s+1} is a complex root, then the quantity ζ_{s+1}^* is determined by formula (4), where one must put $\mu = \mu_{s+1}$, $\gamma = \gamma_{s+1}$. In all other cases $\zeta_{s+1}^* = \zeta_{s+1}$.

Let us formulate the theorem on reduction of order for the closed-loop system shown in Fig. 1. The original and the "truncated"closed-loop systems correspond to the transfer functions of the open-loop systems

$$W(p) = K/pM(p)Q(p); \qquad \widetilde{W}(p) = K/pM(p).$$

The roots of the polynomials $M(p)$ and $Q(p)$ satisfy the conditions given in Theorem 5. By $y_x(t)$ and $z_x(t)$ we denote the output quantities of the original and the "truncated" closed-loop systems, corresponding to zero initial conditions and to the same input action $x(t)$ from the class F. Let

$$\varepsilon_{\max}(t) = \max_{x \in F} |y_x(t) - z_x(t)|.$$

Theorem 6. If the gain coefficient K satisfies the inequalities

$$a^*|M(-a^*)Q(-a^*)| \leqslant K \leqslant \lambda_1 |M(-a^*)Q(-a^*)|a^*,$$

then

$$\varepsilon_{\max}(t) \leqslant (1 - \exp(-\tilde{\gamma}\tilde{a}t))(\tilde{\gamma}\varkappa)^{-1} B(\tilde{\gamma}\tilde{a}) A_{m+1}(\tilde{\gamma}, \tilde{\mu})(1 + H_n(\gamma, \mu_*)).$$

Here $0 < \gamma < 1$ is an arbitrary parameter; $\tilde{a} = a^*|M(-a^*)Q(-a^*)|$; $\gamma = \tilde{\gamma}\tilde{a}/a^*$. The quantities a^*, $A_{m+1}(\tilde{\gamma}, \tilde{\mu})$, $B(\tilde{\gamma}, \tilde{a})$ are determined by the corollary to Theorem 3 and by formulas (3), (12). Applying the corollary to Theorem 4 to the polynomial $pM(p)Q(p)$, we obtain μ_*. The quantity $\tilde{\mu}$ is determined by Theorem 4, if it is applied to the polynomial $pM(p)$ and one puts

$$G = \lambda_1 (1 + \mu_0^2)^{-1/2} \prod_{j=1}^{m} (1 + \mu_j^2)^{1/2}$$

in the right-hand side of (9).

$$H_n(\gamma, \mu_*) = (1 - \gamma)^{-n}, \quad \text{if } \mu_* \leqslant \varphi_1(\gamma).$$

If $\mu_* > \varphi_1(\gamma)$, then

$$H_n(\gamma, \mu_*) = [(1 + \mu_*^2)(2(1 - \gamma)\mu_*)^{-1}]^{n/2} \quad \text{for even } n;$$

$$H_n(\gamma, \mu_x) = [(1 + \mu_x^2)(2(1 - \gamma)\mu_*)^{-1}]^{(n-1)/2}(1 - \gamma)^{-1} \quad \text{for odd } n.$$

Theorems 1–4 and 6 make it possible to relate easily determined parameters of individual elements to the greatest accumulated error of a closed-loop astatic system and to estimate the effect of introducing correcting networks.

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References

1. B. V. Bulgakov, *DAN*, **51**, No. 5 (1946).

2. B. V. Bulgakov, N. T. Kuzovkov, *PMM*, **14**, 1 (1950).

3. Ya. Z. Tsypkin, *Avtomatika i telemekh.*, **13**, No. 4 (1952).

4. Ya. Z. Tsypkin, P. V. Bromberg, *Izv. AN SSSR, OTN*, No. 12 (1945).

5. Z. Sh. Blokh, *Transient Processes in Linear Systems of Automatic Regulation*, 1961.

6. A. M. Rubinshik, *Devices and Elements of the Theory of Automation and Telemechanics*, 1952.

7. A. A. Feldbaum, *Avtomatika i telemekh.*, **9**, No. 4 (1948).

8. N. G. Chebotarev, *Avtomatika i telemekh.*, **9**, No. 4 (1948).

9. E. G. Uderman, *The Method of the Root Hodograph in the Theory of Automatic Control*, 1963.

10. G. A. Bendrikov, K. F. Teodorchik, *Trajectories of Linear Automatic Systems*, 1964.

11. M. V. Meerov, *Multivariable Control Systems*, 1965.

12. A. V. Kalyaev, *Avtomatika i telemekh.*, **20**, No. 9 (1959).

13. A. A. Kardashov, *Avtomatika i telemekh.*, **24**, No. 8 (1963).

Note: Figure translations are in progress. See original paper for figures.