Mathematics

1968

SovietRxiv

Abstract

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Mathematics

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On the Differential Properties of Solutions of Quasielliptic Equations in Unbounded Domains

(Presented by Academician S. L. Sobolev, 3 X 1967)

In the present note we consider solutions of an equation of the form

$$L(\partial/\partial x)U = f, \tag{1}$$

where $L(-is) = L(-is_1, \dots, -is_n)$ is a polynomial with constant coefficients in n variables $(s_1, \dots, s_n) = s$, satisfying the following conditions: a) the polynomial $L(-is)$ does not vanish for real values $s_1, \dots, s_n$ when $|s| \neq 0$; b) the polynomial $L(-is)$ is quasi-homogeneous, i.e., for every $\lambda > 0$ one has

$$L(-is\lambda^{\alpha}) = L(-is_1\lambda^{\alpha_1}, \dots, -is_n\lambda^{\alpha_n}) = \lambda L(-is),$$

where $1/\alpha_i$ are integers. We shall call the vector $\alpha = (\alpha_1, \dots, \alpha_n)$ the homogeneity exponent of the operator L.

Such equations have received in the literature the name quasielliptic and have been well studied locally (see ([1])). The properties of solutions in the large have been investigated much less. The deepest results here have been obtained in terms of Gevrey classes for the case of a homogeneous equation. Among the works in which the inhomogeneous case is considered, we note ([2]), where, under the condition that the polynomial $L(-is)$ does not vanish at any point of the real space R_n (condition b) may fail), a uniqueness class is singled out and the differential properties of solutions are studied in terms of the classes S^{**}, and also in some other classes.

Numerous works have also been devoted to the study of solutions of an equation of the form (1) under various boundary conditions. Among the recent works we mention here ([5–8])***, in which the differential properties of solutions are considered in the Sobolev classes W_2^l.

In the present work, classes of functions are constructed in terms of which, in the whole space R_n, sharp estimates are obtained for solutions of equation (1) in dependence on the differential properties of the function f. The results obtained in the classes under consideration are definitive.

Let us introduce definitions. Let R_n be the Euclidean space of points $x = (x_1, \dots, x_n)$, $R^+_{n+1} = (x \in R_n, x_{n+1} \geq 0)$, $|x| = [\sum x_i^2]^{1/2}$, $x^\beta = x_1^{\beta_1} \cdots x_n^{\beta_n}$,

$$D^\beta = D^{\beta_1} \cdots D^{\beta_n} = \partial^{\beta_1}/\partial x_1^{\beta_1} \cdots \partial^{\beta_n}/\partial x_n^{\beta_n}.$$

We also set

$$R_v[f(x)] = \prod_{i=1}^{n} \frac{1}{v_i} \int_0^{v_1} \cdots \int_0^{v_n} |f(x+u)|\, du,$$

$$\Delta^k(f, he_i) = \sum_{\nu=0}^{k} (-1)^{k-\nu} C_k^\nu f(x_1, \dots, x_{i-1}, x_i + \nu h, x_{i+1}, \dots, x_n),$$

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* For a bibliography of works devoted to uniqueness of solutions of general hypoelliptic equations, see ([3]).

** The definition of the classes and their properties are considered in ([4]).

*** A more complete bibliography of works devoted to quasielliptic equations is given in ([8]).

$$\omega(f, he_i) = \sup_{0<t<h,\ x \in R_n} \left|\Delta^1(f, he_i)\right|, \qquad \rho : r = \sum_{i=1}^{n} \frac{\rho_i}{r_i}.$$

For $1 < p < \infty$, introduce the norms

$$\|f\|_{L_p^l} = \sum_{i=1}^{n} \|D^{l_i} f\|_{L_p(R_n)}; \tag{2}$$

$$\|f\|_{L_p^{l,r}} = \sum_{\rho:r=1} \left(\|D^\rho f\|_{L_p(R_n)} + \|D^\rho f\|_{L_p^l}\right); \tag{3}$$

$$\|f\|_{\mathcal{L}_p^{l,r}} = \|f\|_{L_p^{0,r}} + \sum_{\rho:r=1} \sum_{i=1}^{n} \left(\int_0^\infty h^{-1-l_i p} \|\Delta^s(D^\rho f, he_i)\|_{L_p}^p \, dh\right)^{1/p}, \tag{4}$$

where $s_i > l_i > 0,\ i = 1, 2, \dots, n$.

We shall say that a function f belongs to the Sobolev functional space $L_p^{l,r}(R_n)$ (to the space $\mathcal{L}_p^{l,r}(R_n)$) if f belongs to the closure of the infinitely differentiable finite functions in the norm (3) (in the norm (4)).

Define the weighted space $L_{p,\alpha}^{m,r}(R^+_{n+1})$ as the closure of finite functions in the norm

$$\|f\|_{L_{p,\alpha}^{m,r}} = \sum_{\rho:r=1} \left(\|D^\rho f\|_{L_p} + \sum_{i=1}^{n} \|x_{n+1}^{\alpha_i} D^{\rho+m_i} f\|_{L_p}\right). \tag{5}$$

Then the functional space $\mathcal{L}_p^{l,r}(R_n)$ can also be defined as the space of traces, for $x_{n+1} = 0$, of functions belonging to the weighted space $L_{p,\alpha}^{m,r}(R_{n+1}^+)$ under an appropriate choice of the parameters m and α (see, for example, (9)).

Let us also introduce Hölder classes of functions. Put

$$\|f\|_C = \max_{x\in R_n} |f(x)|,$$

$$\|f\|_{C^\mu} = \sum_{i=1}^n \sup_{1>h>0,\ x\in R_n} \frac{\omega^1(f, he_i)}{h^\mu},$$

$$\|f\|_{C^l} = \sum_{i=1}^n \sum_{0\le k_i\le \bar{l}_i} \|D^{k_i} f\|_{C^\mu},$$

where $\bar{l}_i + \mu = l_i$, $\bar{l}_i$ are integers,

$$\|f\|_{C_\varepsilon^{l,r}(R_n)} = \sum_{\rho:r=1} \|D^\rho f\|_C + \|D^\rho f\|_{C^l} + \sum_{\rho:r=1} \sup_{x\in R_n,\ |v|\ge 0} |v^\varepsilon R_v[D^\rho f(x)]|, \quad (6)$$

where $0 < \varepsilon < \mu < 1$.

Denote by $\mathfrak{M}$ the set of smooth functions for which the norm (6) is finite and, moreover,

$$\overline{\lim_{v\to\infty}} \sup_{x\in R_n} R_v[f(x)] = 0.$$

Then we shall say that f belongs to the space $C^{l,r}(R_n)$ if f belongs to the closure of $\mathfrak{M}$ in the norm (6). Let E_1, E_2 be complete spaces of functions of the form (3)–(6). Define a generalized solution $U \in E_1$ of equation (1) as the limit, in the norm E_1, of solutions $U_\nu \in E_1$

$$Lu_\nu = f_\nu,$$

where f_ν converges to f in the norm E_2.

Theorem 1. Let $f \in L_p^{l,0}(R_n)$, $1 < p < \infty$, and let α be the homogeneity exponent of the operator L. Then there exists a generalized solution U of equation (1) that belongs to the class $L_p^{l,s/\alpha}(R_n)$,

$$c_2\|f\|_{L_p^{l,0}} \le \|U\|_{L_p^{l,1/\alpha}} \le c_1\|f\|_{L_p^{l,0}}.$$

u, for each generalized solution $v \in L_p^{l,1/\alpha}$ there is a representation $V(x) = U(x) + P(x)$, where $P(x)$ is a polynomial of degree not exceeding $1/\alpha_i$ in each variable x_i and such that $D^\rho P(x) \equiv 0$ for all ρ satisfying the condition

$$\sum_{i=1}^n \rho_i\alpha_i = 1.$$

The proof of this theorem relies essentially on the representation formula for the solution of equation (1) for summable functions, obtained by the author. It may be regarded as a generalization, to the case of quasieliptic equations, of John' s well-known formula [10]. Using one interpolation theorem of Lions [11], from this theorem we obtain as a consequence the following result.

Theorem 2. Let $f \in \mathcal{L}^{l,0}(R_n)$, $1 < p < \infty$, and let $\alpha = (\alpha_1, \dots, \alpha_n)$ be the homogeneity exponent of the operator L. Then there exists a generalized solution of equation (1), $U \in \mathcal{L}^{l,1/\alpha}$,

$$c_2 \|f\|_{\mathcal{L}^{l,0}} \le \|U\|_{\mathcal{L}^{l,1/\alpha}} \le c_1 \|f\|_{\mathcal{L}^{l,0}},$$

and for each generalized solution $v \in L_p^{l1/\alpha}$ there is a representation

$$V(x) = U(x) + P(x),$$

where $P(x)$ is a polynomial satisfying the conditions of Theorem 1.

From Theorem 1 one can also obtain an embedding theorem strengthening the corresponding results of S. M. Nikol' skii [12] and V. P. Il' in [13].

Theorem 3. Let $f \in L_p^l(R_n)$, $1 < p < \infty$. Then there exists a polynomial P_l of degrees not exceeding $l_i - 1$ in each variable x_i, such that

$$\|D^\rho (f - P_l)\|_{L_p(R_n)} \le c \|f\|_{L_p^l(R_n)},$$

where c does not depend on f or P_l, and

$$\sum_{i=1}^n \frac{\rho_i}{l_i} = 1.$$

An analogous estimate is also valid for functions $f \in \mathcal{L}_p^l(R_n)$.

Theorem 4. Let $f \in C^l(R_n)$; moreover,

$$\sup_{|\nu| \ge 0,\ x \in R_n} |\nu|^\varepsilon R_\nu[f(x)] < \infty.$$

Then there exists a generalized solution U of equation (1), which belongs to the class $C_{\varepsilon_1}^{l,1/\alpha}$, where $\varepsilon_1 < \varepsilon$, and α is the homogeneity exponent of the operator L, and the estimate holds

$$c_2 \left(\|f\|_{C^l} + \sup_{|\nu| \ge 0,\ x \in R_n} |\nu|^{\varepsilon_1} R_\nu[f] \right) \le \|U\|_{C_{\varepsilon_1}^{l,1/\alpha}} \le c_1 \left(\|f\|_{C^l} + \sup_{|\nu| > 0,\ x \in R_n} |\nu|^\varepsilon R_\nu[f] \right),$$

where the constants c_1, c_2 do not depend on f or U.

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Received
21 IX 1967

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Note: Figure translations are in progress. See original paper for figures.