LIMIT THEOREMS FOR FUNCTIONALS OF MEASURABLE RANDOM PROCESSES

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LIMIT THEOREMS FOR FUNCTIONALS OF MEASURABLE RANDOM PROCESSES

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I. I. Gikhman and A. V. Skorokhod in ([2]) consider, among other things, a very narrow class of functionals (functionals of integral type) of random processes of a very general form and obtain limit theorems for functionals of integral type. It turns out, however, that from limit theorems for functionals of integral type one can obtain limit theorems for a broad class of functionals. The present article is devoted to the study of this question.

Consider a sequence of measurable random processes $\xi_n(t)$, whose finite-dimensional distributions converge to the finite-dimensional distributions of a measurable random process $\xi(t)$ $(t \in [0,1])$.

Suppose there exist $p \geq 1$ and $\eta > 0$ such that

$$\sup_n \sup_t M|\xi_n(t)|^{p+\eta} = c < \infty. \tag{1}$$

Then the sequence of processes $\xi_n(t)$ and the process $\xi(t)$ can be realized on one and the same space of measurable functions X, having the property that if $x(t) \in X$, then

$$\int_0^1 |x(t)|^p \, dt = d < \infty.$$

Introduce on the space X the pseudometric*

$$\rho(x_1, x_2) = \left(\int_0^1 |x_1(t) - x_2(t)|^p \, dt \right)^{1/p}.$$

Every Borel set of the space X will be measurable with respect to the measures $\mu_1, \mu_2, \dots, \mu_n, \dots$ and the measure μ corresponding to the processes $\xi_1(t), \xi_2(t), \dots, \xi_n(t), \dots$ and to the process $\xi(t)$.

Theorem 1. *If for every $\varepsilon > 0$*

$$\lim_{h\to 0} \overline{\lim_{n\to\infty}} \sup_{|t-s|\le h} P\{|\xi_n(t) - \xi_n(s)| > \varepsilon\} = 0 \tag{2}$$

and our processes satisfy the listed conditions (measurability, convergence of finite-dimensional distributions, equality (1)), then

$$\lim_{n\to\infty} \int f(x)\, d\mu_n(x) = \int f(x)\, d\mu(x), \tag{3}$$

where $f(x)$ is an arbitrary continuous functional on X.

Proof. We perform the natural mapping of the space X onto the metric space X, which is a subset of L_p. On

* A function of two variables on X, $\rho(x_1, x_2)$, is called a pseudometric if it satisfies the following properties: 1) $\rho(x_1, x_2) = \rho(x_2, x_1) \ge 0$; 2) $\rho(x_1, x_2) + \rho(x_2, x_3) \ge \rho(x_1, x_3)$.

to the space $\hat{X}$ the measures $\mu_1, \mu_2, \ldots, \mu_n, \ldots$ and the measure μ, which on $\hat{X}$ we shall denote respectively by $\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_n, \ldots$ and $\tilde{\mu}$. The proof of the theorem will follow from the fact that

$$\lim_{n\to\infty} \int f(x)\, d\tilde{\mu}_n(x) = \int f(x)\, d\tilde{\mu}(x) \tag{4}$$

for every continuous and bounded functional on $\hat{X}$. The proof of equality (4) will be preceded by the following lemma.

Lemma. *Let measures $\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_n, \ldots$ and a measure $\tilde{\mu}$ be given on the Borel sets of a metric space $\hat{X}$. Suppose there exists a separable ring $G(X)$ of continuous and bounded functions on $\hat{X}$, separating any point and any closed set not containing that point and possessing the property that, if $g(x) \in G(X)$, then*

$$\lim_{n\to\infty} \int g(x)\, d\tilde{\mu}_n(x) = \int g(x)\, d\tilde{\mu}(x). \tag{5}$$

Then equality (4) follows from equality (5).

Proof of the lemma. The ring $G(\hat{X})$ determines a compact extension of the space $\hat{X}$–$g\hat{X}$. Transfer to the compact $g\hat{X}$ the measures $\tilde{\mu}_1, \tilde{\mu}_2, \ldots, \tilde{\mu}_n, \ldots$ and the measure $\tilde{\mu}$, which we denote respectively by $\bar{\mu}_1, \bar{\mu}_2, \ldots, \bar{\mu}_n, \ldots$ and $\bar{\mu}$. For any function $f(x)$ continuous on $g\hat{X}$ the equality

$$\lim_{n\to\infty} \int f(x)\, d\bar{\mu}_n(x) = \int f(x)\, d\bar{\mu}(x)$$

holds.

If U is an open set in $g\hat{X}$ having the property that $\bar{\mu}([U]) = \bar{\mu}(U)$, then (see (1))

$$\lim_{n\to\infty} \bar{\mu}_n(U) = \bar{\mu}(U), \tag{6}$$

where $[U]$ is the closure of U in $g\hat{X}$. From equality (6) for the compact $g\hat{X}$ the analogous equality follows for the space $\hat{X}$, which proves the lemma.

Let us construct the ring $G(X)$. The generators of this ring will be the functionals

$$h(x) = \min\left(\int_0^1 |x(t) - a(t)|^p\, dt,\ q\right),$$

where $a(t)$ are polynomials with rational coefficients, and q are rational numbers.

To prove equality (5), observe that, by virtue of the results of I. I. Gikhman and A. V. Skorokhod ([2]), the distributions of the functionals

$$\int_0^1 \sum_{k=1}^{l} \lambda_k |\xi_n(t) - a_k(t)|^p\, dt$$

converge to the distribution of the functional

$$\int_0^1 \sum_{k=1}^{l} \lambda_k |\xi(t) - a_k(t)|^p\, dt,$$

where $a_1(t), a_2(t), \dots, a_k(t), \dots, a_l(t)$ are arbitrary continuous functions, and $\lambda_1, \lambda_2, \dots, \lambda_k, \dots, \lambda_l$ are arbitrary constants (the proof of I. I. Gikhman and A. V. Skorokhod uses the measurability of the random processes, the convergence of finite-dimensional distributions, and also equalities (1) and (2)).

Theorem 1 is proved.

Let us pass to the study of bounded processes.

Theorem 2. *Suppose the finite-dimensional distributions of measurable random processes $\xi_n(t)$ converge to the finite-dimensional distributions of a measurable random process $\xi(t)$ $(t \in [0,1])$, and for all n, t, $0 \le \xi_n(t) \le$*

$\leqslant 1$. Then, for an arbitrary continuous and bounded functional $f(x)$ (p may be any number greater than or equal to 1), equality (3) holds.

Proof. The sequence of processes $\xi_n(t)$ and the process $\xi(t)$ can be realized on one and the same space of measurable functions X, possessing the property that, if $x(t) \in X$, then $0 \leqslant x(t) \leqslant 1$. On the space X there exists a ring of continuous and bounded functionals of integral form

$$F(x(t)) = \int_0^1 \cdots \int_0^1 \varphi(t_1, \dots, t_q; x(t_1), \dots, x(t_q))\, dt_1 \cdots dt_q,$$

where $\varphi(t_1, \dots, t_q; x_1, \dots, x_q)$ is a continuous function on the unit cube of dimension $2q$. But, since the processes are measurable,

$$\begin{aligned}\lim_{n\to\infty} MF(\xi_n(t)) &= \lim_{n\to\infty} M \int_0^1 \cdots \int_0^1 \varphi(t_1, \dots, t_q; \xi_n(t_1), \dots, \xi_n(t_q))\, dt_1 \cdots dt_q \\ &= \lim_{n\to\infty} \int_0^1 \cdots \int_0^1 M\varphi(t_1, \dots, t_q; \xi_n(t_1), \dots, \xi_n(t_q))\, dt_1 \cdots dt_q \\ &= \int_0^1 \cdots \int_0^1 M\varphi(t_1, \dots, t_q; \xi(t_1), \dots, \xi(t_q))\, dt_1 \cdots dt_q \\ &= M \int_0^1 \cdots \int_0^1 \varphi(t_1, \dots, t_q; \xi(t_1), \dots, \xi(t_q))\, dt_1 \cdots dt_q = MF(\xi(t)).\end{aligned}$$

On the space $\hat{X}$ the ring of functionals of integral form separates any point and any closed set not containing that point. The condition of the lemma is fulfilled. Theorem 2 is proved.

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Note: Figure translations are in progress. See original paper for figures.