REGULARIZATION ALGORITHMS FOR LINEAR EQUATIONS WITH UNBOUNDED OPERATORS

MATHEMATICS

1968

SovietRxiv

Abstract

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UDC 517.948.35

MATHEMATICS

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REGULARIZATION ALGORITHMS FOR LINEAR EQUATIONS WITH UNBOUNDED OPERATORS

(Presented by Academician A. N. Tikhonov on 4 III 1968)

Consider the operator equation:

$$Ax = f, \tag{1}$$

where A is a linear closed operator with an everywhere dense domain of definition in some Hilbert space H_1, acting into the Hilbert space H_2, having no bounded inverse, and $f \in A(D_A)$. The spaces H_1 and H_2 are assumed to be complete and real.

Interest in the approximate solution of such equations has recently increased thanks to the works of A. N. Tikhonov (for example ([1],[2])). In these works the problem of solving (1) was reduced to the problem of constructing a regularizing algorithm. This reduction proved convenient, in particular, for the numerical solution of equation (1).

In what follows, by a regularizing algorithm (r.a.) for equation (1) we shall mean a family of linear bounded operators $\{R\}$, acting from H_2 to H_1, such that for any $f \in A(D_A)$ one can indicate $x \in A^{-1}f$ such that for any $\varepsilon > 0$ there exists $\delta = \delta(f, \varepsilon, A)$ and an operator R from our family such that $\|R\tilde{f} - x\| \le \varepsilon$, provided only that $\|\tilde{f} - f\| \le \delta$.

In ([3]) an abstract analogue of the r.a. considered in ([2]) was extended to operator equations (1), under the assumption that $H_1 = H_2$ and $f \in D(A^*)$. The aim of the present work is to generalize the scheme for obtaining an r.a. proposed in our work ([4]) for bounded operators A, to equations with unbounded operators. The r.a. studied in ([1–3]) enters into this scheme as a special case.

First of all, let us note that, by the theorem of J. von Neumann ([5]), there exists a self-adjoint operator A^*A with dense domain of definition $D(A^*A)$, acting from H_1 to H_1. Denote by $\{E_\lambda\}$ its resolution of the identity.

Let the real function $\psi(\lambda,\alpha)$ possess the following properties: it is defined for $\lambda \in S(A^*A)$ and $\alpha > 0$ as follows:

$$\psi(\lambda,\alpha) = \begin{cases} \varphi(\lambda,\alpha)/\lambda, & \lambda \neq 0, \\ K, \quad |K| < \infty, & \lambda = 0. \end{cases} \tag{2}$$

$\varphi(\lambda,\alpha)$ is bounded with respect to λ and α, measurable in λ for each α with respect to $\{E_\lambda\}$; moreover,

a) $\lim\limits_{\alpha\to 0} \varphi(\lambda,\alpha) = 1, \quad \varphi(0,\alpha) = 0;$

b) $\sup\limits_{\substack{\lambda \in S(A^*A) \\ \lambda \neq 0}} |\varphi(\lambda,\alpha)|/\sqrt{\lambda} = K_\alpha < \infty \quad (\alpha > 0),$

where in a) the convergence to the limit is uniform on the set $S(A^*A) \cap (c,\infty)$, where c is any positive number.

Denote

$$\Omega_k = \{\lambda :\ k-1 \leq \psi(\lambda,\alpha) < k\}, \quad k = \dots, -2, -1, 0, 1, \dots$$

By our assumptions, every set Ω_k is measurable with respect to $\{E_\lambda\}$.

Let $E(\Omega_k)$ be its spectral measure. Define the operator $B_{k\alpha}$ by the formula

$$B_{k\alpha} = \overline{\int_{\Omega_k} \psi(\lambda,\alpha)\, dE_\lambda E(\Omega_k) A^*}. \tag{3}$$

(The bar denotes the closure of an operator. The possibility of taking the closure will be shown below.)

The operator $B_{k\alpha}$ is bounded and $\|B_{k\alpha}\| \leq K_\alpha$. Indeed, the operator

$$\int_{\Omega_k} \psi(\lambda,\alpha)\, dE_\lambda E(\Omega_k) A^*$$

is defined on an everywhere dense set in H_2 and has an adjoint given by the formula

$$AE(\Omega_k) \int_{\Omega_k} \psi(\lambda,\alpha)\, dE_\lambda = A \int_{\Omega_k} \psi(\lambda,\alpha)\, dE_\lambda E(\Omega_k). \tag{4}$$

Moreover, on a dense set in H_1 the equality

$$\left(AE(\Omega_k)\int_{\Omega_k}\psi(\lambda,\alpha)\,dE_\lambda x,\ AE(\Omega_k)\int_{\Omega_k}\psi(\lambda,\alpha)\,dE_\lambda x\right)=\left(\int_{\Omega_k}\lambda\psi^2(\lambda,\alpha)\,dE_\lambda x,\ x\right) \tag{5}$$

holds.

Indeed, if $x\in\overline{H}_{1k}=E(\Omega_k)H_1$, then equality (5) is valid (both sides are equal to 0). If $x\in H_{1k}$, then the operator

$$\int_{\Omega_k}\psi(\lambda,\alpha)\,dE_\lambda$$

is invertible in the subspace H_{1k} ($k\neq 0,1$). In the case $k=0$ (or 1) the space H_{1k} itself can be decomposed into a direct sum of its invariant subspaces with respect to

$$\int_{\Omega_k}\psi(\lambda,\alpha)\,dE_\lambda,$$

in each of which the operator

$$\int_{\Omega_k}\psi(\lambda,\alpha)\,dE_\lambda$$

is invertible. Consequently, since $D(A^*A)$ is dense in H_{1k}, there exists a dense set in H_{1k} (respectively, in each of the subspaces forming H_{1k}) on which the operator

$$\int_{\Omega_k}\psi(\lambda,\alpha)\,dE_\lambda E(\Omega_k)A^*AE(\Omega_k)\int_{\Omega_k}\psi(\lambda,\alpha)\,dE_\lambda=\int_{\Omega_k}\lambda\psi^2(\lambda,\alpha)\,dE_\lambda$$

is defined.

From (5) and the closedness of the operator (4) it follows that (4) is bounded; moreover, by condition b) its norm is $\leq K_\alpha$. The adjoint of (4) is equal to $B_{k\alpha}$. Therefore, $\|B_{k\alpha}\|\leq K_\alpha$. Finally, note that the ranges of the operators $B_{k\alpha}$ for different k are orthogonal. Define the operator

$$B_\alpha=\sum_{k=-\infty}^{\infty}\int_{\Omega_k}\psi(\lambda,\alpha)\,dE_\lambda E(\Omega_k)A^*. \tag{6}$$

By what was said above, the operator (6) is bounded. It is easy to show that the exact equality $\|B_\alpha\|=K_\alpha$ holds.

We shall now prove that the family $\{B_\alpha\}$ forms a r.a. for equation (1). Let $f\in A(D_A)$. Take such a solution $\hat{x}$ of (1) for which the condition $\hat{x}\perp\ker A^*A$

is satisfied. This means that the function $(E_\lambda \hat{x}, \hat{x})$ is continuous at $\lambda = 0$. Note that

$$B_\alpha A\hat{x} = \int_0^\infty \varphi(\lambda, \alpha)\, dE_\lambda \hat{x}.$$

Consider $B_\alpha \tilde{f} - \hat{x}$. We have

$$\|B_\alpha \tilde{f} - \hat{x}\| \leq \|B_\alpha A\hat{x} - \hat{x}\| + K_\alpha \|\tilde{f} - f\| =$$

$$= \left(\int_0^\infty (\varphi(\lambda, \alpha) - 1)^2\, d(E_\lambda \hat{x}, \hat{x}) \right)^{1/2} + K_\alpha \|\tilde{f} - f\|. \tag{7}$$

By virtue of condition a) and the continuity of $(E_\lambda \hat{x}, \hat{x})$ at $\lambda = 0$, for any $\varepsilon > 0$ one can choose such an α (and consequently also B_α) and such a δ that $\|B_\alpha \tilde{f} - \hat{x}\| \leqslant \varepsilon$, provided only that $\|\tilde{f} - f\| \leqslant \delta$. All the specific functions $\psi(\lambda, \alpha)$ (with the exception of $\psi(\lambda, \alpha) = [1 - (1 - \mu\lambda)^{1/\alpha}]/\lambda$ for $\lambda \neq 0$; $\psi(\lambda, \alpha) = \mu/\lambda$ for $\lambda = 0$), considered in ([4]), are suitable for constructing r.a. in our case. In particular, for (1) one can also construct an "optimal" r.a. ([4]). In the case when $H_1 = H_2$ and A is self-adjoint, the scheme for obtaining r.a. can be substantially simplified. The results of ([4]) carry over to equation (1) in this case without any changes.

An analogous scheme for obtaining r.a. in this special case was also proposed in ([6]).

The r.a. obtained above may be applied, for example, to constructing an approximate solution of unstable boundary-value problems for differential equations under the condition that such a problem is solvable, and in other analogous cases.

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Received
1 III 1968

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Note: Figure translations are in progress. See original paper for figures.