THE ALGEBRA OF VOLTERRA OPERATORS AND ITS APPLICATION IN PROBLEMS OF VISCOELASTICITY

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Abstract

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THE ALGEBRA OF VOLTERRA OPERATORS AND ITS APPLICATION IN PROBLEMS OF VISCOELASTICITY

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1. To solve boundary-value problems of linear viscoelasticity based on continuous spectra of relaxation or retardation times, it is necessary to construct an algebra of Volterra operators. The first such attempt was made by Yu. N. Rabotnov (ˆ1). For a special class of operators with fractional-exponential kernels, the basic facts of the algebra were obtained. However, the desire to describe more fully the viscoelastic properties of real bodies leads to operators of a more complex nature ([2]-4). Often, for such operators even the construction of the resolvent encounters insurmountable difficulties (ˆ3), and all the more the question of solving boundary-value problems on their basis remains open.

We shall show that the construction of an algebra of Volterra operators is not connected with any special form of them and can be carried out for arbitrary resolvent operators. Therefore the realization in time of solutions of boundary-value problems of viscoelasticity within the framework of Volterra' s principle is completely algebraized, i.e., can be carried out without any concretization of the initial operators.

2. Let P^* be a Volterra integral operator:

$$P^*(...) = \int_0^t P(t-\tau)(...)\,d\tau; \tag{1}$$

$P(t)$ is the kernel of the operator. If λ is a regular point (ˆ5), then the operator

$$\overline{R}_\lambda = (I-\lambda P^*)^{-1} = I + \lambda R_\lambda^* \tag{2}$$

is called the inverse, and the operator R_λ^* is the resolvent for P^*.

Every invertible operator P^* generates a one-parameter family of resolvent operators R_λ^* $(-\infty < \lambda < \infty)$. The general algorithm for constructing these operators is the Neumann series (ˆ5)

$$R_\lambda^* = \sum_{k=0}^{\infty} \lambda^k P^{*\,k+1}. \tag{3}$$

On the basis of representation (3) of the resolvent operator, one can prove the following theorem:

Theorem 1 (multiplication). *For any two regular points λ and μ the equality holds*

$$R_\lambda^* R_\mu^* = \frac{1}{\lambda - \mu}\,(R_\lambda^* - R_\mu^*). \tag{4}$$

The proof follows from the following chain of equalities:

$$R_\lambda^* - R_\mu^* = \sum_{k=0}^{\infty} (\lambda^k - \mu^k) P^{*\,k+1} = (\lambda - \mu) \sum_{k=0}^{\infty} \sum_{l=0}^{k-1} \lambda^{k-1-l} \mu^l P^{*\,k+1} =$$

$$= (\lambda - \mu) \left(\sum_{m=0}^{\infty} \lambda^m P^{*\,m+1} \right) \left(\sum_{n=0}^{\infty} \mu^n P^{*\,n+1} \right) = (\lambda - \mu) R_\lambda^* R_\mu^*.$$

This theorem generalizes the theorem of Yu. N. Rabotnov on the multiplication of fractional-exponential operators ([1]) and is an operator analogue of the functional relation for resolvents ([5,6]).

A consequence of the multiplication theorem is

Theorem 2 (inversion). *For any resolvent operator R_λ^* the inversion formula holds*

$$(I - \varkappa R_\mu^*)^{-1} = I + \varkappa R_{\mu+\varkappa}^*, \tag{5}$$

i.e., inversion of a resolvent operator reduces to a shift in the parameter.

The set of regular points of the operator P^* is the continuum $(-\infty, \infty)$; therefore it makes sense to consider the values of products of resolvent operators when these points approach one another without bound. From (4) we find:

$$R_\lambda^{*2} = \partial R_\lambda^* / \partial \lambda. \tag{6}$$

By the method of induction one can prove the following theorem:

Theorem 3 (on powers). *Raising a resolvent operator to a power is equivalent to differentiation with respect to the parameter:*

$$R_\lambda^{*n} = \frac{1}{(n-1)!}\,\frac{\partial^{\,n-1} R_\lambda^*}{\partial\lambda^{\,n-1}}. \tag{7}$$

For fractional-exponential operators this theorem was proved in ([7]). As a consequence of this theorem we obtain that the Neumann series in the inversion of a resolvent operator is identical to the Taylor series:

$$R_{\lambda+\mu}^* = \sum_{k=0}^{\infty} \mu^k R_\lambda^{*k+1} = \sum_{k=0}^{\infty} \frac{\mu^k}{k!}\,\frac{\partial^k R_\lambda^*}{\partial\lambda^k}, \tag{8}$$

i.e., iteration of kernels in raising to a power is equivalent to their differentiation with respect to the parameter. This is often more convenient in concrete constructions.

The theorems of M. I. Rozovskii ([8],[9]) for fractional-exponential operators carry over completely to arbitrary resolvent operators.

Theorem 4. *If* $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$, *then*

$$\prod_{k=1}^{n} R_{\lambda_k}^* = \sum_{k=1}^{n} \left[\prod_{\substack{i=1\\ i\neq k}}^{n} (\lambda_k - \lambda_i)\right]^{-1} R_{\lambda_k}^*. \tag{9}$$

Theorem 5. *If* $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$, *then*

$$\left[1 - \sum_{k=1}^{n} M_k \prod_{i=1}^{k} R_{\lambda_i}^*\right]^{-1} = 1 + \sum_{k=1}^{n} a_k R_{\mu_k}^*, \tag{10}$$

The parameters $a_k,\ \mu_k$ are found in the same way as in (8).

Theorem 6. *If* $\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$, *then*

$$\prod_{i=1}^{n} R_{\lambda_i}^{*m_i} = \left[\prod_{i=1}^{n} (m_i - 1)!\right]^{-1} \frac{\partial^{m_1+m_2+\cdots+m_n-n}}{\partial\lambda_1^{m_1-1}\partial\lambda_2^{m_2-1}\cdots\partial\lambda_m^{m_n-1}} \times$$

$$\times \left\{\sum_{i=1}^{n} \left[\prod_{k=1}^{n} (\lambda_i - \lambda_k)\right]^{-1} R_{\lambda_i}^*\right\} \quad (i \neq k). \tag{11}$$

These theorems prove useful in constructing the theory of anisotropic viscoelastic media.

3. As is known ([10]), functions of operators can be defined in various ways. The results given make it possible to implement the simplest approach, generalizing the concept of a power of an operator. Although in this way

only analytic functions of resolvent operators can be defined; for applications this proves sufficient.

The simplest examples may be rational and fractional-rational functions of operators:

$$M\left[R_\lambda^*\right] = \sum_{k=0}^{n} a_k R_\lambda^{*k} = \sum_{k=0}^{n} \frac{a_k}{(k-1)!}\,\frac{\partial^{k-1} R_\lambda^*}{\partial \lambda^{k-1}} = M_0 \prod_{k=1}^{n} (I - \mu_k R_\lambda^*), \qquad (12)$$

$$S\left[R_\lambda^*\right] = \frac{N\left[R_\lambda^*\right]}{M\left[R_\lambda^*\right]} = \frac{\sum\limits_{k=0}^{m} b_k R_\lambda^{*k}}{\sum\limits_{k=0}^{n} a_k R_\lambda^{*k}} = \sum_{k=0}^{m-n} c_k R_\lambda^{*k} - \sum_{k=0}^{n} \mu_k c_{-k} (I + \mu_k R_{\lambda+\mu_k}^*), \quad (13)$$

where $M_0 = (-1)^n a_0 z_1 z_2 \dots z_n$; $\mu_k = 1/z_k$; z_k are simple, nonzero real roots of the polynomial $M(z)$; c_k, c_{-k} are the coefficients of the expansion of the operator fraction (13) into simplest fractions. The regular part in (13) exists if $m-n \geqslant 1$.

Generalizations of the functions (12) and (13) are, respectively, entire and meromorphic functions of the resolvent operator. If $T(z)$ is an entire function, then the operator $T[R_\lambda^*]$ will be called regular. Under the condition of boundedness (even weak singularity) of the kernel R_λ^*, this operator has a completely definite meaning on the set of summable functions, since the series expressing the action of the operator on a summable function converges for every finite value of time. If $S(z)$ is a meromorphic function, then the operator $S[R_\lambda^*]$ is called singular. Using the Mittag-Leffler representation (11) for a meromorphic function, one can give a definition of such an operator that generalizes (13).

The most general class of functions of resolvent operators admitting a simple realization are the analytic functions

$$F\left[R_\lambda^*\right] = \sum_{k=0}^{\infty} a_k R_\lambda^{*k} = \sum_{k=0}^{\infty} \frac{a_k}{(k-1)!}\,\frac{\partial^{k-1} R_\lambda^*}{\partial \lambda^{k-1}}. \qquad (14)$$

If the series $\sum_{k=0}^{\infty} a_k z^k$ converges in some neighborhood of zero, then the series

$$\sum_{k=0}^{\infty} a_k R_\lambda^{*k}(f)$$

for a summable function $f(t)$ converges for every finite time. The sum of this series is the value of the operator. An example of such an operator is the irrational operator

$$(I - \mu R_\lambda^*)^p = I - p\mu R_\lambda^* + \frac{p(p-1)}{2}\mu^2 \frac{\partial R_\lambda^*}{\partial \lambda} + \dots \tag{15}$$

A particular case of the operator (15) for $p = 1/2$, with a fractional-exponential kernel (1), was applied in (12) to the study of creep of mine-working contours.

What has been said above, like Theorems 4, 5, 6, is readily generalized to compositional analytic functions of resolvent operators of the form

$$F\left[R_{\lambda_1}^*, R_{\lambda_2}^*, \dots, R_{\lambda_n}^*\right] = \sum_{k=0}^{\infty} A_k \prod_{i=1}^{n} R_{\lambda_i}^{*m_{ki}}, \tag{16}$$

which is essential for constructing a theory of anisotropic viscoelasticity.

4. A common feature of the integral operators of viscoelasticity is their resolvent character, which follows from the postulate of the unambiguous mutual reversibility of the stress-strain relations. From this fact and the preceding results it follows that, in solving boundary-value problems of visco-

in elasticity there is no need to construct in advance the specific form of the resolvent and to study the properties of the operators. The procedure for constructing the solution according to Volterra' s principle is connected only with the algebraic properties of the operators and therefore can always be carried out independently of their specifics. The solution is expressed through the values of the resolvent operator and its derivatives at certain regular points (including zero). Finding these points is the principal stage in realizing the solution. They are determined by certain sets of rheological parameters, depending only on the form of the operator function being realized.

The results presented make it possible to realize a fairly broad class of functions of viscoelasticity operators that arise in solving problems. The specific values of these operators are required only at the stage of analyzing the solution obtained. They can always be found with the necessary degree of accuracy by using approximate algorithms. This significantly broadens the possibility of choosing integral operators for describing the phenomena of isotropic and anisotropic viscoelasticity.

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Note: Figure translations are in progress. See original paper for figures.