INTEGRABLE CASES IN THE PROBLEM OF THE MOTION OF A HEAVY RIGID BODY IN A FLUID

HYDROMECHANICS

1968

SovietRxiv

Abstract

Full Text

UDC 531.338

HYDROMECHANICS

V. N. RUBANOVSKII

INTEGRABLE CASES IN THE PROBLEM OF THE MOTION OF A HEAVY RIGID BODY IN A FLUID

(Presented by Academician L. I. Sedov, 17 VII 1967)

We consider a dynamical system consisting of an infinitely extended incompressible ideal fluid and a rigid body moving in it, bounded by a simply connected surface and having multiply connected cavities of arbitrary shape, completely filled with an ideal fluid performing vortex-free motion.

It is assumed that the body and the fluids are acted upon by gravity, the weight of the fluid displaced by the body being equal to the weight of the body and of the fluids in its cavities ([1]).

We shall suppose that the kinetic energy of such a dynamical system is written in the form ([2])

$$T = \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(a_{ij}P_{iP}j + b_{ij}R_{iR}j + 2c_{ij}P_{iR}j\right), \qquad a_{ij} = a_{ji}, \quad b_{ij} = b_{ji},$$

where a_{ij}, b_{ij}, c_{ij} are constants determined for the given system, and P_i, R_i are the projections on the axes of the rectangular coordinate system $Ox_1x_2x_3$, rigidly attached to the body, of the impulsive force $\mathbf{R}$ and the impulsive couple $\mathbf{P}$ of the system, without taking into account the cyclic motion of the fluid in the cavities of the body.

Assuming that the impulsive force is directed along the line of action of gravity, we shall have the following equations of motion of a heavy rigid body in a fluid under the conditions of S. A. Chaplygin ([2]-[4]):

$$\begin{gathered} dR_1/dt + R_3\partial T/\partial P_2 - R_2\partial T/\partial P_3 = 0 \quad (123), \\ dP_1/dt + (P_3+g_3)\partial T/\partial P_2 - (P_2+g_2)\partial T/\partial P_3 + \\ +R_3\partial T/\partial R_2 - R_2\partial T/\partial R_3 = r_2R_3 - r_3R_2 \quad (123). \end{gathered} \tag{1}$$

Here $\mathbf{r}(r_1, r_2, r_3)$ is a vector proportional to the radius vector drawn from the center of gravity of the volume bounded by the surface of the body in contact with the infinite fluid to the center of gravity of the body and of the fluid contained in its cavities, and $\mathbf{g}(g_1, g_2, g_3)$ is the vector of the kinetic moment with respect to the origin of the moving axes of the cyclic motion of the fluid in the cavities of the body.

The following three first integrals of equations (1) are known ([3], [4]):

$$T - r_1 R_1 - r_2 R_2 - r_3 R_3 = \text{const},$$

$$(P_1 + g_1)R_1 + (P_2 + g_2)R_2 + (P_3 + g_3)R_3 = \text{const},$$

$$R_1^2 + R_2^2 + R_3^2 = \text{const}.$$

The following assertions are valid.

1. If the kinetic energy of the system has the form

$$T = \frac{a}{2}\left(P_1^2 + P_2^2 + P_3^2\right) + \frac{(c_{33} - c_{22})^2}{2a} R_1^2 + \frac{(c_{11} - c_{33})^2}{2a} R_2^2 + \frac{(c_{22} - c_{11})^2}{2a} R_3^2 + \\ + c_{11} P_1 R_1 + c_{22} P_2 R_2 + c_{33} P_3 R_3 + \frac{b}{2}\left(R_1^2 + R_2^2 + R_3^2\right),$$

and the vector $\mathbf{g} = 0$, then equations (1) admit a fourth first integral

$$c_{11}\left[P_1 + \frac{c_{22} + c_{33}}{a} R_1 - \frac{r_1}{2c_{11}}\right]^2 + c_{22}\left[P_2 + \frac{c_{33} + c_{11}}{a} R_2 - \frac{r_2}{2c_{22}}\right]^2 +$$

$$+ c_{33}\left[P_3 + \frac{c_{11} + c_{22}}{a} R_3 - \frac{r_3}{2c_{33}}\right]^2 = \text{const}.$$

In particular, for $r_3 = 0$ the case of P. V. Kharlamov ([5]) obtains, and for $r_1 = r_2 = r_3 = 0$ the case of A. M. Lyapunov ([6]).

2. If

$$T = a_{11}P_1^2 + a_{22}P_2^2 + a_{33}P_3^2 + 2\sigma(a_{22}a_{33}P_1R_1 + a_{33}a_{11}P_2R_2 + a_{11}a_{22}P_3R_3) +$$

$$+ \sigma^2[a_{11}(a_{22}^2 + a_{33}^2)R_1^2 + a_{22}(a_{33}^2 + a_{11}^2)R_2^2 + a_{33}(a_{11}^2 + a_{22}^2)R_3^2] +$$

$$+ 2c(P_1R_1 + P_2R_2 + P_3R_3) + b(R_1^2 + R_2^2 + R_3^2),$$

and the vectors $\mathbf{r}$ and $\mathbf{g}$ are proportional,

$$\mathbf{r} = -[c + \sigma(a_{11}a_{22} + a_{22}a_{33} + a_{33}a_{11})]\mathbf{g},$$

then equations (1) admit a fourth first integral

$$P_1^2 + P_2^2 + P_3^2 - 2\sigma(a_{11}P_1R_1 + a_{22}P_2R_2 + a_{33}P_3R_3) +$$

$$+ \sigma^2[(a_{33} - a_{22})^2R_1^2 + (a_{11} - a_{33})^2R_2^2 + (a_{22} - a_{11})^2R_3^2] +$$

$$+ 2\sigma(a_{11}g_1R_1 + a_{22}g_2R_2 + a_{33}g_3R_3) + 2(g_1P_1 + g_2P_2 + g_3P_3) = \text{const.}$$

In particular, for $g_3 = 0$ the case of P. V. Kharlamov ([3]) obtains, for $g_1 = g_2 = g_3 = 0$ the case of V. A. Steklov ([7]), and for $\sigma = 0$ the case of N. E. Zhukovsky ([8]), when equations (1) reduce to the equations of motion of a balanced gyroscope.

3. Let

$$2T = a_{11}P_1^2 + a_{22}P_2^2 + a_{33}P_3^2 + 2a_{12}P_1P_2 +$$

$$+ 2\left[c_{33} + (a_{11} - a_{33})\frac{c_{21}}{a_{12}} + a_{33}\frac{c_{12}}{a_{12}}\right]P_1R_1 +$$

$$+ 2\left[c_{33} + (a_{22} - a_{33})\frac{c_{12}}{a_{12}} + a_{33}\frac{c_{21}}{a_{12}}\right]P_2R_2 + 2c_{33}P_3R_3 + 2c_{12}P_1R_2 + 2c_{21}P_2R_1 +$$

$$+ a_{11}\frac{c_{21}^2}{a_{12}^2}R_1^2 + a_{22}\frac{c_{12}^2}{a_{12}^2}R_2^2 + a_{33}\frac{(c_{12} - c_{21})^2}{a_{12}^2}R_3^2 +$$

$$+ 2\frac{c_{12}c_{21}}{a_{12}}R_1R_2 + b(R_1^2 + R_2^2 + R_3^2),$$

and the vectors $\mathbf{r}$ and $\mathbf{g}$ are perpendicular to the axis ox_3, with

$$r_1 = \left[-c_{33} + (c_{12} + c_{21})\frac{a_{33}}{a_{12}} + 2\frac{c_{12}a_{33}^2}{a_{12}\Delta}(a_{33} - a_{22})\right]g_1 + 2\frac{c_{12}a_{33}^2}{\Delta}g_2,$$

$$r_2 = \left[-c_{33} + (c_{12} + c_{21})\frac{a_{33}}{a_{12}} + 2\frac{c_{21}a_{33}^2}{a_{12}\Delta}(a_{33} - a_{11})\right] g_2 + 2\frac{c_{21}a_{33}^2}{\Delta}g_1,$$

$$\Delta = a_{12}^2 - (a_{11} - a_{33})(a_{22} - a_{33}).$$

Then equations (1) admit the aggregate system of linear integrals

$$P_1 + \frac{c_{21}}{a_{12}}R_1 = \frac{a_{33}}{\Delta}[(a_{33} - a_{22})g_1 + a_{12}g_2],$$

$$P_2 + \frac{c_{12}}{a_{12}}R_2 = \frac{a_{33}}{\Delta}[(a_{33} - a_{11})g_2 + a_{12}g_1],$$

$$P_3 + \frac{c_{12} + c_{21}}{a_{12}}R_3 = \text{const}.$$

In particular, for $g_1 = g_2 = 0$ the case of S. A. Chaplygin ([9]) obtains.

4. If the kinetic energy of the system is determined by the formula

$$2T = a_{11}P_1^2 + a_{22}P_2^2 + a_{33}P_3^2 + 2a_{12}P_1P_2 + 2c_{11}P_1R_1 + 2c_{22}P_2R_2 + 2c_{33}P_3R_3+$$

$$+ 2c_{12}P_1R_2 + 2c_{21}P_2R_1 + a_{11}\frac{c_{21}^2}{a_{12}^2}R_1^2 + a_{22}\frac{c_{12}^2}{a_{12}^2}R_2^2+$$

$$+ a_{33}\frac{(c_{12} - c_{21})^2}{a_{12}^2}R_3^2 + 2\frac{c_{12}c_{21}}{a_{12}}R_1R_2 + b(R_1^2 + R_2^2 + R_3^2),$$

where

$$a_{12}^2 = (a_{11} - a_{33})(a_{22} - a_{33}), \quad c_{11} = c_{33} + (a_{11} - a_{33})\frac{c_{21}}{a_{12}} + a_{33}\frac{c_{12}}{a_{12}},$$

$$c_{22} = c_{33} + (a_{22} - a_{33})\frac{c_{12}}{a_{12}} + a_{33}\frac{c_{21}}{a_{12}},$$

and the vectors $\mathbf{r}$ and $\mathbf{g}$ are perpendicular to the axis Ox_3, with

$$(a_{11} - a_{33})\frac{a_{12}^2}{c_{12}^2}\left\{r_1 + \left[c_{33} - (c_{12} + c_{21})\frac{a_{33}}{a_{12}}\right] g_1\right\}^2 -$$

$$-\frac{2a_{12}a_{33}^2}{c_{12}}\left\{r_1+\left[c_{33}-(c_{12}+c_{21})\frac{a_{33}}{a_{12}}\right]g_1\right\}g_1=$$

$$=(a_{22}-a_{33})\frac{a_{12}^2}{c_{21}^2}\left\{r_2+\left[c_{33}-(c_{12}+c_{21})\frac{a_{33}}{a_{12}}\right]g_2\right\}^2-$$

$$-\frac{2a_{12}a_{33}^2}{c_{21}}\left\{r_2+\left[c_{33}-(c_{12}+c_{21})\frac{a_{33}}{a_{12}}\right]g_2\right\}g_2,$$

then equations (1) admit the following simultaneous system of linear integrals:

$$P_1+\frac{c_{21}}{a_{12}}R_1=-\frac{a_{12}}{2c_{12}a_{33}}\left\{r_1+\left[c_{33}-(c_{12}+c_{21})\frac{a_{33}}{a_{12}}\right]g_1\right\},$$

$$P_2+\frac{c_{12}}{a_{12}}R_2=-\frac{a_{12}}{2c_{21}a_{33}}\left\{r_2+\left[c_{33}-(c_{12}+c_{21})\frac{a_{33}}{a_{12}}\right]g_2\right\},$$

$$P_3+\frac{c_{12}+c_{21}}{a_{12}}R_3=\text{const.}$$

In particular, when $g_1=g_2=0$, the case of P. V. Kharlamov ([5]) is obtained.

5. Let

$$\begin{aligned}2T={}&a_{11}P_1^2+a_{22}P_2^2+a_{33}P_3^2+2a_{12}P_1P_2+2a_{23}P_2P_3+2a_{13}P_3P_1+\\&+2c_{11}P_1R_1+2c_{22}P_2R_2+2c_{33}P_3R_3+2a_{12}k_2P_1R_2+2a_{12}k_1P_2R_1+\\&+2a_{23}k_3P_2R_3+2a_{23}k_2P_3R_2+2a_{13}k_1P_3R_1+2a_{13}k_3P_1R_3+\\&+b_{11}R_1^2+b_{22}R_2^2+b_{33}R_3^2+2a_{12}k_1k_2R_1R_2+2a_{23}k_2k_3R_2R_3+2a_{13}k_3k_1R_3R_1,\end{aligned}$$

where

$$c_{11}=\mu+(a_{11}+\lambda)k_1\ (123),\qquad b_{11}=\sigma+\lambda k_2k_3+(a_{11}+\lambda)k_1^2\ (123),$$

and the projections of the vectors $\mathbf{r}$ and $\mathbf{g}$ are related by the relations

$$r_1=\frac{\lambda}{2}(k_1-k_2-k_3)(s_1+g_1)-(\mu+\lambda k_1)g_1\ (123),$$

$$s_1=\frac{\chi}{\Delta(\chi)}\left\{[(a_{22}-\chi)(a_{33}-\chi)+a_{23}^2]g_1+[a_{13}a_{23}-a_{12}(a_{33}-\chi)]g_2+\right.$$

$$+[a_{12}a_{23}-a_{13}(a_{22}-\chi)]g_3\} \ \ (123), \tag{2}$$

$$\Delta(\chi)=(a_{11}-\chi)(a_{22}-\chi)(a_{33}-\chi)-$$

$$-(a_{11}-\chi)a_{23}^2-(a_{22}-\chi)a_{13}^2-(a_{33}-\chi)a_{12}^2, \qquad \chi=-\lambda/2.$$

Then equations (1) admit the simultaneous system of linear integrals

$$P_1+k_1R_1=s_1, \qquad P_2+k_2R_2=s_2, \qquad P_3+k_3R_3=s_3. \tag{3}$$

When $g_1=g_2=g_3=0$, the case of S. A. Chaplygin ([9]) is obtained.

6. If the kinetic energy of the system is

$$2T=a_{11}P_1^2+a_{22}P_2^2+a_{33}P_3^2+2a_{12}P_1P_2+2a_{23}P_2P_3+2a_{13}P_3P_1+$$

$$+2c(P_1R_1+P_2R_2+P_3R_3)+2a_{12}k_2P_1R_2+2a_{12}k_1P_2R_1+$$

$$+2a_{23}k_3P_2R_3+2a_{23}k_2P_3R_2+2a_{13}k_1P_3R_1+2a_{13}k_3P_1R_3+$$
$$+(b+\tau a_{22}a_{33})R_1^2+(b+\tau a_{33}a_{11})R_2^2+(b+\tau a_{11}a_{22})R_3^2+2a_{12}k_1k_2R_1R_2+$$
$$2a_{23}k_2k_3R_2R_3+2a_{13}k_3k_1R_3R_1,$$

where

$$k_1=\varepsilon\sqrt{(\sigma+\tau a_{22})(\sigma+\tau a_{33})}/\sqrt{\sigma+\tau a_{11}} \quad (123)$$
$$(\varepsilon=\pm 1;\ \sigma,\tau \text{ arbitrary}),$$

and the projections of the vectors $\mathbf{r}$ and $\mathbf{g}$ are related by the relations

$$r_1=\frac{1}{2\tau}[2k_1k_2k_3-\sigma(k_1+k_2+k_3)](s_1+g_1)-$$
$$-\frac{1}{\tau}k_1k_2k_3s_1+\frac{\sigma}{\tau}k_1s_1-cg_1 \quad (123),$$

in which the quantities s_1, s_2, s_3 are determined by formulas (2) for $\chi=-\sigma/2\tau$, then equations (1) admit the aggregate system of linear integrals (3).

For $a_{12}=a_{23}=a_{13}=0$ and $g_1=g_2=g_3=0$, the case of V. A. Steklov[10] obtains.

7. If

$$2T = a(P_1^2 + P_2^2 + P_3^2) + 2a_{12}P_1P_2 + 2a_{23}P_2P_3 + 2a_{13}P_3P_1 +$$
$$+ 2c(P_1R_1 + P_2R_2 + P_3R_3) + 2a_{12}k_2P_1R_2 + 2a_{12}k_1P_2R_1 + 2a_{23}k_3P_2R_3 +$$
$$+ 2a_{23}k_2P_3R_2 + 2a_{13}k_1P_3R_1 + 2a_{13}k_3P_1R_3 + b_{11}R_1^2 + b_{22}R_2^2 + b_{33}R_3^2 +$$
$$+ 2a_{12}k_1k_2R_1R_2 + 2a_{23}k_2k_3R_2R_3 + 2a_{13}k_3k_1R_3R_1,$$

where

$$k_1 = \varepsilon\sqrt{(\sigma - b_{22})(\sigma - b_{33})}/\sqrt{a(\sigma - b_{11})} \quad (123)$$
$$(\varepsilon = \pm 1;\ \sigma \text{ arbitrary}),$$

and the projections of the vectors **r** and **g** are related by the relations

$$2r_1 = a(k_2 + k_3 - k_1)(s_1 + g_1) + (c - ak_1)g_1 \quad (123),$$

in which the quantities s_1, s_2, s_3 are determined by formulas (2) for $\chi = a/2$, then equations (1) admit the aggregate system of linear integrals (3).

8. Let the kinetic energy of the system be given by the expression

$$2T = a(P_1^2 + P_2^2 + P_3^2) + 2a_{12}P_1P_2 + 2a_{23}P_2P_3 + 2a_{13}P_3P_1 +$$
$$+ 2c_{11}P_1R_1 + 2c_{22}P_2R_2 + 2c_{33}P_3R_3 + 2a_{12}k_2P_1R_2 + 2a_{12}k_1P_2R_1 +$$
$$+ 2a_{23}k_3P_2R_3 + 2a_{23}k_2P_3R_2 + 2a_{13}k_1P_3R_1 + 2a_{13}k_3P_1R_3 + b(R_1^2 + R_2^2 + R_3^2) +$$
$$+ a_{12}k_1k_2R_1R_2 + a_{23}k_2k_3R_2R_3 + a_{13}k_3k_1R_3R_1,$$

where

$$k_1 = \frac{2}{3a}(c_{22} + c_{33} - 2c_{11}) \quad (123),$$

and the projections of the vectors **r** and **g** are related by the relations

$$3r_1 = (c_{22} + c_{33} - 2c_{11})s_1 - (c_{11} + c_{22} + c_{33})g_1 \quad (123),$$

in which the quantities s_1, s_2, s_3 are determined by formulas (2) for $\chi = a/4$. Then equations (1) admit the aggregate system of linear integrals (3).

For $a_{12} = a_{23} = a_{13} = 0$ and $g_1 = g_2 = g_3 = 0$, the case of V. A. Steklov[10] obtains.

Moscow State University
named after M. V. Lomonosov

Received
7 VII 1967

CITED LITERATURE

[1] S. A. Chaplygin, A new particular solution of the problem of the motion of a rigid body in a fluid, Collected Works, 1, Publ. House of the Academy of Sciences of the USSR, 1933, p. 151.
[2] A. Clebsch, Math. Ann., 3, No. 2, 238 (1870).
[3] P. V. Kharlamov, Appl. Math. and Tech. Phys., No. 4, 17 (1963).
[4] G. R. Kirchhoff, J. f. reine u. angew. Math., 71, 237 (1869).
[5] P. V. Kharlamov, DAN, 107, No. 3, 381 (1956).
[6] A. M. Lyapunov, Collected Works, 1, Publ. House of the Academy of Sciences of the USSR, 1954, p. 320.
[7] W. Stekloff, Math. Ann., 42, 273 (1893).
[8] N. E. Zhukovskii, Collected Works, 3, M.–L., 1936, pp. 21–187.
[9] S. A. Chaplygin, On certain cases of motion of a rigid body in a fluid, Article 2, Collected Works, 1, Publ. House of the Academy of Sciences of the USSR, 1933, p. 43.
[10] V. A. Steklov, Proceedings of the Physics Section of the Society of Lovers of Natural Science, 7, issue 2, 1895.

Note: Figure translations are in progress. See original paper for figures.