ON THE RIEMANN PROBLEM IN DOMAINS WHOSE BOUNDARY HAS BOUNDED ROTATION

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Abstract

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MATHEMATICS

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ON THE RIEMANN PROBLEM IN DOMAINS WHOSE BOUNDARY HAS BOUNDED ROTATION

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1. Let a closed rectifiable Jordan curve $\Gamma : t = t(s)$, $0 \leq s \leq S$, have no cusps, and let the angle of its tangent with the abscissa axis be representable in the form $\theta(s) = \theta_{0,1}(s) + \theta_2(s)$, $0 \leq s \leq S$, where $\theta_{0,1}(s)$ is a function of bounded variation on $[0, S]$, and $\theta_2(s)$ satisfies the Hölder condition with exponent $0 < \alpha \leq 1$. In particular, when $\theta_2(s) \equiv 0$ we have a curve of bounded rotation (see (^{1}), p. 98), and when $\theta_{0,1}(s) \equiv 0$, a Lyapunov curve. Denote by D^+, D^- the finite and infinite domains of the complex z-plane bounded by this curve.

In the article (^{2}) it was proved that the singular integral operator

$$Kf(\tau) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)}{t - \tau}\, dt, \qquad \tau \in \Gamma, \tag{1}$$

is bounded in L_p, $p > 1$. In the present paper our aim is to prove its boundedness in the space L_p with a certain weight and to investigate in the domains $D^\pm$ the Riemann boundary-value problem with discontinuous coefficients.

Suppose that on Γ a function $G(t) = |G(t)|e^{i\varphi(t)}$ is defined, possessing the following properties: 1) $0 < m \leq |G(t)| \leq M < \infty$ almost everywhere on Γ; 2) the function $\varphi(t)$ is continuous on Γ everywhere except for at most a countable set of points $\{t_k = t(s_k)\}_{k=1}^{\infty} \neq t_0 = t(0)$, where it has jumps h_k, not exceeding π in absolute value, and the point t_0, where it has the jump $-2\pi\chi_0$ (χ_0 is an integer); 3)

$$\sum_{k=1}^{\infty} |h_k| < \infty$$

(whence it follows that the jumps h_k may be regarded as ordered in nonincreasing order of their absolute values); 4) $h_k \neq 2\pi/p$, if $h_k > 0$, and $|h_k| \neq 2\pi/p'$, if $h_k < 0$, $1/p + 1/p' = 1$.

It is not difficult to show that, having prescribed an arbitrary $\varepsilon > 0$, one can find such a number n that the decomposition

$$\varphi(t) = \psi_n(t) + \omega_n(t),$$

is valid, in which the function $\psi_n(t)$ has jumps h_k at the points t_k, $k = 1, 2, \dots, n$, and the jump $-2\pi\chi_0$ at the point t_0, and on the arcs between these points satisfies the Lipschitz condition; while the function $\omega_n(t)$ satisfies the inequality

$$\sup_{t\in\Gamma} |\omega_n(t)| < \varepsilon.$$

Consider the Riemann problem in the following formulation.

Find a function $\Phi^+(z) \in E_p(D^+)$ (see (^{3}), p. 203), and a function $\Phi^-(z) \in E_p(D^-)$, vanishing at infinity, which almost everywhere on Γ satisfy the boundary condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \tag{2}$$

where $g(t) \in L_p(\Gamma)$.

Problem (2) in the classical formulation, when $G(t)$ and $g(t)$ satisfy the Hölder condition and Γ is a Lyapunov curve, was solved by F. D. Gakhov (^{4}). For its further generalizations see (^{4},^{5}). In the indicated formulation and under assumptions on the coefficients, problem (2), in the case where Γ is the unit circle, was studied by I. I. Danilyuk (^{6}). The method of computing the index used below belongs to him. In similar assumptions, problems for Lyapunov curves were studied by I. B. Simonenko (^{7}).

2. Let

$$X^\pm(z) = \exp\left\{\frac{1}{2\pi i}\int_\Gamma \frac{\ln|G(t)| + i\varphi(t)}{t - z}\,dt\right\}, \qquad z \in D^\pm.$$

From the Sokhotski formulas it follows that $G(t) = X^+(t)/X^-(t)$, $t \in \Gamma$. Let us study to which classes $X^\pm(z)$ belong.

First of all, note that the functions

$$\left[\exp\frac{1}{2\pi i}\int_\Gamma \frac{\ln|G(t)|}{t - z}\,dt\right]^{\pm 1}$$

are bounded in $D^\pm + \Gamma$, since, as can be proved using results of I. Radon (see (^{1}), p. 102), the real part of the integral occurring here has this property.

Now consider

$$W^{\pm}(z) = \exp\left\{\frac{1}{2\pi}\int_{\Gamma}\frac{\varphi(t)}{t-z}\,dt\right\}, \qquad z \in D^{\pm}.$$

Lemma. For sufficiently small $\delta > 0$, the functions $W^{\pm}(z)$ belong to the classes $E_{\delta}(D^{\pm})$.

We shall carry out the proof for D^{+}. Construct a sequence of functions $\{\varphi_k(t)\}_{k=1}^{\infty}$, $\sup_{t\in\Gamma,k}|\varphi_k(t)| < \infty$, satisfying the Lipschitz condition on Γ and converging in measure to $\varphi(t)$. The functions

$$W_k^{+}(z) = \exp\left\{\frac{1}{2\pi}\int_{\Gamma}\frac{\varphi_k(t)}{t-z}\,dt\right\}$$

are continuous in $D^{+}+\Gamma$ (see (^{3}), p. 197). Further, there exists a constant M, independent of k, such that

$$\int_{\Gamma}|W_k^{+}(t)|^{\delta}\,ds \le M.$$

This follows from Cauchy' s formula applied to the function $[W_k^{+}(z)]^{\delta}\exp i\Phi_0(z)$, where $\Phi_0(z)$ is a function continuous in $D^{+}+\Gamma$ such that, for sufficiently small $\delta > 0$,

$$\sup_{t\in\Gamma,k}\left|\delta(\varphi_k(t)/2 + \operatorname{Re} K\varphi_k(t)) + (\theta(t) - \arg t) + \operatorname{Re}\Phi_0(t)\right| < \pi/2$$

(see (^{2})). Denote by γ_r the image of the circle $|\zeta| = r$ under the conformal mapping $z = w(\zeta)$ of the disk $|\zeta| < 1$ onto D^{+}. Then

$$\int_{\gamma_r}|W_k^{+}(z)|^{\delta}|dz| = r\int_0^{2\pi}|W_k^{+}(w(re^{i\alpha}))|\sqrt[\delta]{\rho_{\delta}'}\,d\alpha \le$$

$$\le \int_0^{2\pi}|W_k^{+}(w(e^{i\alpha}))|\sqrt[\delta]{w'(e^{i\alpha})}\,d\alpha = \int_{\Gamma}|W_k^{+}(t)|^{\delta}\,dt \le M$$

(see (^{3}), pp. 78, 89). Since $W^{+}(z) = \lim_{k\to\infty}W_k^{+}(z)$ for $z \in D^{+}$, the proof is completed by applying Fatou' s lemma. It is obvious that an analogous assertion is true for $[W^{\pm}(z)]^{-1}$. We also note that, by virtue of the property of the function $\omega_n(t)$, for arbitrary $1 < q < \infty$ one can indi-

choose a number n such that

$$[\Omega_n(z)]^{\pm1} = \left[\exp\left\{\frac{1}{2\pi}\int_{\Gamma}\frac{\omega_n(t)}{t-z}\,dt\right\}\right]^{\pm1} \in E_q(D^{\pm}).$$

3. Let $h_1^+ \geq h_2^+ \geq \cdots$ be all the positive jumps of the function $\varphi(t)$, and let $h_1^- \geq h_2^- \geq \cdots$ be the moduli of the negative jumps, $\{t_k^+\}_{k=1}^{\infty}$, $\{t_k^-\}_{k=1}^{\infty}$ the corresponding subsets of Γ. Define the numbers $\varkappa^+$, $\varkappa^-$ by the following conditions:

$$\varkappa^+ = \max\{k :\ h_k^+ > 2\pi/p\}; \qquad \varkappa^- = \max\{k :\ h_k^- > 2\pi/p'\}.$$

Since the singular integral has a logarithmic singularity at a jump point of the density, it is not difficult to show that

$$\Psi_n(\tau) \equiv |\tau - t_0|^{-\varkappa_0} \left| \frac{\prod\limits_{k=1}^{\varkappa^+} |\tau - t_k^+|}{\prod\limits_{k=1}^{\varkappa^-} |\tau - t_k^-|} \right| \exp\{iK\psi_n(\tau)\} =$$

$$= \frac{\prod\limits_{k=1}^{\varkappa^+} |\tau - t_k^+|^{1-h_k^+/2\pi} \prod\limits_{k=\varkappa^-+1}^{n_1} |\tau - t_k^-|^{h_k^-/2\pi}}{\prod\limits_{k=1}^{\varkappa^-} |\tau - t_k^-|^{1-h_k^-/2\pi} \prod\limits_{k=\varkappa^++1}^{n_2} |\tau - t_k^+|^{h_k^+/2\pi}} |U_n(\tau)|, \tag{3}$$

where

$$\sup_{\tau\in\Gamma} |U_n(\tau)|^{\pm 1} < \infty, \qquad n_1 > \varkappa^-, \quad n_2 > \varkappa^+, \quad n_1 + n_2 = n.$$

For sufficiently small $\lambda > 0$ we have $\Psi_n(\tau) \in L_{p+\lambda}(\Gamma)$, $[\Psi_n(\tau)]^{-1} \in L_{p'+\lambda}(\Gamma)$. Hence, from the lemma it follows that for some $\delta > 0$

$$Z^{\pm}(z) = (z - t_0)^{-\varkappa_0} \frac{\prod\limits_{k=1}^{\varkappa^+} (z - t_k^+)}{\prod\limits_{k=1}^{\varkappa^-} (z - t_k^-)} X^{\pm}(z) \in E_\delta(D^{\pm})$$

and the boundary values of these functions on Γ are summable to the power $p+\lambda_1$, $\lambda_1 < \lambda$. The ratio of the length of the larger of the two arcs determined by an arbitrary chord to the length of this chord is bounded above and, consequently, D^+, D^- are domains of class C (see ([8])). Therefore, by V. I. Smirnov' s theorem (see ([3]), p. 264) $Z^{\pm}(z) \in E_{p+\lambda_1}(D^{\pm})$. Similarly we obtain that $[Z^{\pm}(z)]^{-1} \in E_{p'+\lambda_1}(D^{\pm})$.

4. Using the boundedness of the operator (1) in $L_p(\Gamma)$, $p > 1$, and B. V. Khvedelidze' s argument (see ([9]), p. 24, Theorem 5), we obtain from (3):

$$\|(Kf)\Psi_n\|_{L_p} \leq M_1 \|f\Psi_n\|_{L_p}.$$

Let $1 < q < \infty$, $0 < c < 1$; $\Omega_n(\varepsilon) = |\Omega_n^+(\tau)|$, $\tau \in \Gamma$. Consider the problem

$$\Phi^+(t) = ce^{i\omega_n(t)}\Phi^-(t) + g(t). \tag{4}$$

Here $g(t) \in L_q(\Gamma)$, and the solutions are sought in $E_q(D^\pm)$.

Given arbitrary $\eta > 0$, one can choose the numbers c and n so that

$$\sup_{t\in\Gamma} |ce^{i\omega_n(t)} - 1| < \eta, \qquad [\Omega_n^\pm(z)]^{-1} \in E_q(D^\pm).$$

Then, from the results of I. B. Simonenko (see (^{10}); (^{7}), p. 286), transferred to the case of our domains, it follows that problem (4) is uniquely and unconditionally solvable, and therefore

$$\|(Kf)\Omega_n\|_{L_q} \leq M_2 \|f\Omega_n\|_{L_q}.$$

We now choose numbers $\gamma > 1$ and $\varepsilon > 1$ so that $\gamma h_k^+ \neq 2\pi/(p-\varepsilon)$, $\gamma h_k^- \neq 2\pi/(p-\varepsilon)$, $k = 1, 2, ...$, and put $p_1 = p - \varepsilon (< k)$,

$$p_2 = \frac{p(p-\varepsilon)(\gamma-1)}{p(\gamma-1)-\gamma\varepsilon}(> p), \quad 0 < t = \frac{\gamma-1}{\gamma} < 1, \quad \delta = \frac{1}{t}.$$

Then, for some sufficiently large n,

$$\|(Kf)\Psi_n^\gamma\|_{L_{p_1}} \leq M_3 \|f\Psi_n^\gamma\|_{L_{p_1}}; \qquad \|(Kf)\Omega_n^\delta\|_{L_{p_2}} \leq M_4 \|f\Omega_n^\delta\|_{L_{p_2}}.$$

Since $p^{-1} = (1-t)p_1^{-1} + tp_2^{-1}$, $\Psi_n^{\gamma(1-t)}\Omega_n^{\delta t} = \Psi_n\Omega_n$, it follows immediately, by Stein' s interpolation theorem (see (^{11})):

Theorem 1. *The operator*

$$\frac{Z^+(\tau)}{2\pi i}\int_\Gamma \frac{g(t)}{Z^+(t)(t-\tau)}\,dt, \qquad \tau \in \Gamma,$$

is bounded in the space $L_p(\Gamma)$, $p > 1$.

5. The results obtained make it possible to analyze problem (2):

Theorem 2. *We shall call the number*

$$\varkappa = \varkappa_0 - \varkappa^+ + \varkappa^-$$

the index of problem (2).

Then, for $\varkappa \geq 0$, the problem is unconditionally solvable and has $\varkappa$ linearly independent solutions. In the case $\varkappa < 0$, the problem is solvable only when $|\varkappa|$ conditions are satisfied:

$$\int_\Gamma \frac{g(t)}{Z^+(t)}\, t^k\, dt = 0, \qquad k = 0, 1, \ldots, |\varkappa| - 1,$$

and in this case it has a unique solution.

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Note: Figure translations are in progress. See original paper for figures.

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