REALIZATION OF WHITEHEAD TORSION

MATHEMATICS

1968

SovietRxiv

Abstract

Full Text

UDC 513.83

MATHEMATICS

V. L. GOLO

REALIZATION OF WHITEHEAD TORSION AND DISCRIMINANTS OF BILINEAR FORMS

(Presented by Academician P. S. Aleksandrov on 26 IV 1968)

Let there be given a map $f : M_1^{2k-1} \to M_2^{2k-1}$ which is a homotopy equivalence of closed odd-dimensional manifolds of some class. What values of the Whitehead torsion can then arise? The following variants of the formulation of the problem are possible, corresponding to different restrictions on the class of maps f: (1) no additional restrictions are imposed on the homotopy equivalence f (the general realization problem); (2) the map f induces an isomorphism of tangent bundles; (2a) the map f induces an isomorphism of tangent bundles and, moreover, M_1^{2k-1} and M_2^{2k-1} have a common normal invariant (relative to f) in the sense of ([1]); (3) the map f is induced by an h-cobordism W joining M_1 and M_2.

The case (3) was studied earlier, namely, for an h-cobordism the torsion $t(f)$ takes values in the subgroup of the Whitehead group $Wh(\pi_1)$ which is the image of the operator $1 + D$, where D is the duality operator. For the Whitehead torsion under an h-cobordism there is the relation $t(W, \dot{M}_1) = -Dt(W, \dot{M}_2)$ ([3]), and therefore $t(M_1, M_2) = (1 + D)t(W, \dot{M}_1)$, where $Dt = (-1)^{\dim W}\bar{t}$. The conjugation $t \to \bar{t}$ in the group $Wh(\pi_1)$ is obtained by transposing a matrix in $GL(\mathbb{Z}\pi_1)$ and replacing each element $\sum ag_a$ of the matrix by $\sum ag_a^{-1}$. For finite abelian groups the conjugation $t \to \bar{t}$ is trivial, i.e. $t = \bar{t}$ ([3]), and $\operatorname{Im}(1 + D)$ is equal to 0 when $\dim M \equiv 0 \pmod 2$, or to the subgroup of elements of the form $2x$ when $\dim M \equiv 1 \pmod 2$. Thus, by h-cobordisms alone one cannot realize the whole group $Wh(\pi_1)$—there remains the group $Wh(\pi_1)/\operatorname{Im}(1 + D)$.

In most cases the formulations (2) and (2a) turn out to be equivalent; for example, this is so for $\pi_1 = G$, where G is a finite group whose order is not divisible by small primes q (for $q < n/2$). This fact is almost obvious. We shall solve the realization problem in the class of maps (2a) having a trivial normal invariant. Here the solution of the question turns out to be complete. In a number of examples it gives a complete solution of the realization problem.

If $f : M_1^{2k-1} \to M_2^{2k-1}$ is such that the manifolds M_1^{2k-1}, M_2^{2k-1} have one and the same normal invariant, then ([1]) there exists a manifold W^{2k} with two boundary components M_1^{2k-1} and M_2^{2k-1} and two retractions $r_1 : W^{2k} \to M_1^{2k-1}$, $r_2 : W^{2k} \to M_2^{2k-1}$, preserving stable tangent bundles. It is known ([1]) that W^{2k} can be chosen so that the $\mathbb{Z}\pi_1$-module $H_*(\widehat{W}, \widehat{M}_1)$ of the homology of the pair $(\widehat{W}, \widehat{M}_1)$ (by the sign ^ we shall everywhere denote the universal covering) is arranged as follows:

$$H_i(\widehat{W}, \widehat{M}_1) = 0, \qquad \text{if } i \neq k,$$

$$\widehat{H}_k(\widehat{W}, \widehat{M}_1) = H \text{ is free.}$$

Poincaré duality induces on H a nonsingular unimodular bilinear form $\varphi(x, y)$ with values in $\pi_1(1, {}^4)$ such that:

$$(1) \;\; \varphi \text{ is Hermitian, i.e. } \varphi(x, y) = \overline{\varphi(y, x)}, \text{ if } k \text{ is even;}$$

$$(1a) \;\; \varphi \text{ is skew-Hermitian, i.e. } \varphi(x, y) = -\overline{\varphi(y, x)}, \text{ if } k \text{ is odd;}$$

$$(2) \;\; \varphi(x, x) = y + \overline{y}, \text{ if } k \text{ is even;}$$

$$(2a) \;\; \varphi(x, x) = y - \overline{y}, \text{ if } k \text{ is odd.}$$

For a nonsingular unimodular bilinear form one can define invariantly the discriminant ([4]) with values in the quotient of $Wh(\pi_1)$ by the subgroup generated by elements of the form $t + \bar{t}$, i.e. in our case ($\dim W \equiv 0 \bmod 2$) in the group $Wh(\pi_1)/\operatorname{Im}(1 + D)$. To define this discriminant, take an arbitrary free basis of the module H. We obtain the matrix A of the form φ in this basis. Under a change of basis the matrix A passes to QAQ^*, where Q^* is obtained from Q by transposition and by transforming each element $\sum a g_a$ into $\sum a g_a^{-1}$. Consequently, the discriminant of φ, equal modulo $(t + \bar{t})$ to the class of A in $Wh(\pi_1)$, is invariantly defined in $Wh(\pi_1)/\operatorname{Im}(1 + D)$.

Theorem 1. *If the map f induces an isomorphism of tangent bundles and the normal invariant of f is trivial, then*

$$t(f) \equiv \Delta \quad \text{in } Wh(\pi_1)/\operatorname{Im}(1 + D),$$

where Δ is the discriminant of the form φ of the manifold W^{2k}.

We indicate the scheme of the proof of Theorem 1. Consider the sequence of maps of complexes:

$$0 \to \widehat{M}_1 \to \widehat{W} \to (\widehat{W}, \widehat{M}_1) \to H \to 0.$$

Separate out the kernel

$$\operatorname{Ker}\{\widehat{W} \to (\widehat{W}, \widehat{M}_1) \to H\} = C.$$

Proposition 1. *(1) C is a complex of finitely generated free Z_{π_1}-modules; C is embedded in the complex $\widehat{W}$; (2) there exist embeddings $j_1 : \widehat{M}_1 \to C$, $j_2 : \widehat{M}_2 \to C$, inducing a homotopy equivalence of complexes; (3) there exist retractions $\rho_1 : C \to \widehat{M}_1$, $\rho_2 : C \to \widehat{M}_2$, induced by retractions r_1 and r_2; (4) in the complex C one can choose a basis B such that*

$$t(\widehat{M}_1 \xrightarrow{j_1} C) = t(j_1(\widehat{M}_1), C), \qquad t(\widehat{M}_2 \xrightarrow{j_2} C) = t(j_2(\widehat{M}_2), C).$$

Poincaré duality for the complex $\widehat{W}$ gives

Proposition 2. *The Whitehead torsions of the pairs $(C, \widehat{M}_1)$ and $(C, \widehat{M}_2)$ are connected by the relation*

$$t(C, \widehat{M}_1) + Dt(C, \widehat{M}_2) \equiv \Delta(\varphi) \quad \text{in } Wh(\pi_1)/\operatorname{Im}(1 + D).$$

This relation is proved, in essence, in the same way as the usual duality for Whitehead torsion, but one must take into account that the complex C is not a "genuine" geometric complex, but is obtained from W by splitting off the homology module.

From Proposition 2 Theorem 1 now follows. Indeed,

$$t(f) = t(\rho_2 \cdot j_1) = t(\rho_2) + t(j_1).$$

Since ρ_2 is a retraction, $t(\rho_2) = -t(C, \widehat{M}_2)$, whence

$$t(f) = t(C, \widehat{M}_1) - t(C, \widehat{M}_2) = -(1 + D)t(C, \widehat{M}_2) + \Delta,$$

which gives $t(f) \equiv \Delta(\varphi)$ in $Wh(\pi_1)/\operatorname{Im}(1 + D)$.

Apparently, nowhere in the literature is it indicated which subgroups in $Wh(\pi_1)/\operatorname{Im}(1 + D)$ can generate determinants of forms satisfying conditions (1), (2) or (1a), (2a). For Hermitian forms one can immediately

to specify the self-conjugacy condition $\Delta = \bar{\Delta}$, but it is trivially satisfied in the whole group $Wh(\pi_1)$ for a sufficiently broad class of groups (for example, for all finite abelian groups).

S. P. Novikov communicated to me the following simple

Theorem 2. *For any unimodular form φ satisfying conditions (1), (2) (or (1a), (2a)), there exists a manifold W^{2k}, one of whose boundary components is M_1, and φ is a form on the module $H_k(W, M_1)$.*

S. P. Novikov also owns the following

Remark. If the dimension of the manifold M is equal to 7 or 15, then one can construct the manifold W for any Hermitian form φ. This follows from the existence in these dimensions of the Hopf invariant I. From this remark one immediately obtains

Theorem 2a. *If an element $x \in Wh(\pi_1)/\operatorname{Im}(1+D)$ is realized by the discriminant of a unimodular Hermitian form, then it is also realized by the discriminant of a unimodular Hermitian form satisfying condition (2).*

We now give an example in which the discriminants of Hermitian (as well as skew-Hermitian) forms generate a subgroup of units U of the group ring $\mathbb{Z}\pi_1$, where π_1 is finite abelian. As is known, $U \subset Wh(\pi_1)$. Let us note that, generally speaking, $U \neq Wh(\pi_1)$ ([2]).

Example. Let π_1 be the cyclic group of order 5 with generator g. A generator of the group of units of the ring $\mathbb{Z}\pi_1$ is

$$\varepsilon = g + g^{-1}, \qquad \varepsilon \neq 0 \quad \text{in } Wh(\pi_1)/\operatorname{Im}(1+D) \quad ({}^3).$$

Hermitian and skew-Hermitian matrices with determinant $\pm\varepsilon$ are respectively the following:

$$\begin{pmatrix} 2 & g^4 + g^2 + 1 \\ g^{-4} + g^{-2} + 1 & g^2 + g^{-2} + 2 \end{pmatrix}, \quad \begin{pmatrix} g^2 - g^{-2} & -1 \\ 1 & g^2 - g^{-2} \end{pmatrix}.$$

The author expresses his gratitude to S. P. Novikov for his attention to this work.

Mechanical-Mathematical Faculty
of Moscow State University
named after M. V. Lomonosov

Received
26 IV 1968

References

[1] S. P. Novikov, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **28**, 2, 365 (1964).
[2] H. Bass, J. Milnor, J. P. Serre, *Preprint Solution of the Congruence Subgroup Problem*, Columbia University, N. Y., 1966.

[3] J. Milnor, *Preprint Whitehead Torsion*, Princeton University, Princeton, 1964.
[4] C. T. C. Wall, *Ann. Math.*, **84**, No. 2, 217 (1966).

Note: Figure translations are in progress. See original paper for figures.