ON THE THEORY OF INTEGRAL REPRESENTATIONS OF HOLOMORPHIC FUNCTIONS

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ON THE THEORY OF INTEGRAL REPRESENTATIONS OF HOLOMORPHIC FUNCTIONS

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The author, in the case of one and of several complex variables, has established in ($^{1-5}$) a number of general integral representations for holomorphic functions.

In the present note we shall dwell on certain questions in the theory of integral representations of holomorphic functions (mainly in the case of bounded convex complete multicircular domains). In the exposition we adhere to the definitions and notation used in ($^{1-5}$).

1. A. A. Temlyakov ($^{6-9}$) obtained integral representations of two kinds, now known in the mathematical literature (see, for example, (^{10})) as Temlyakov integral representations of the first and second kinds. For a long time the Temlyakov integral representations of the first and second kinds were regarded not as a single Temlyakov integral formula consisting of these representations, but as isolated integral representations (although they were connected by the corresponding relation). Their extension to the case $n > 2$ was likewise carried out in isolation. L. A. Aizenberg (^{11,10}), in extending them, obtained two isolated integral representations (analogues of the first and second kinds), and Li Che Gon (^{12}) obtained one (an analogue of the first kind). In 1963 Opyal and Sityak (^{13}) gave a natural extension of Temlyakov' s integral formula to the case $n > 2$. They obtained an integral formula consisting of n integral representations, which, however, was still far from being a complete extension of Temlyakov' s integral formula to the case $n > 2$. The latter was explained by the fact that the operator nature of Temlyakov' s integral formula had not been fully revealed. As is known, the Temlyakov integral representation of the first kind expresses the values of the function $f(z_1, z_2)$ inside the domain through the values of the operator $L_1[f]$ on the boundary (or on a part of the boundary) of the domain, i.e., it is operatorial with respect to the function f. Only this aspect was used by Opyal and Sityak (^{13}), although the operator nature of Temlyakov' s integral formula proved to be considerably richer. I observed (^{2}), $\S 4, item 2$

that Temlyakov' s integral formula is operatorial simultaneously both with respect to the function f and with respect to the Cauchy kernel

$$\frac{1}{\xi - u}.$$

Here, for any $k = 0, 1, \ldots, \mu$ $(0 \leq \mu \leq n-1)$, the sum of the orders of the operators in this formula is equal to $n-1$, i.e., for a given number of complex variables it remains constant **. In this connection, when considering Temlyakov' s integral formula in the case $n > 2$, the question arose of all possible redistributions in the integrand of the last inner integral of this formula of the operators with Cauchy kernel

$$\frac{1}{\xi - u}$$

onto the function f and conversely, i.e., the problem arose of the complete extension of Temlyakov' s integral formula to the case $n > 2$.

The formula (7.3) obtained by the author in [2] (in the author' s note [1] this is formula (3) for $\alpha = 0$) also solves this problem. Similarly, the formulas (8.3) and (9.2) established by the author in [2] are solutions of the problems of complete extension of the integral formulas of Poisson–Temlyakov and Schwarz–Temlyakov, respectively, to the case $n > 2$.

2. The disclosure of the operator nature of Temlyakov' s integral formula also made it possible for the author [1, 2] to discover, in the case $n = 2$, Temlyakov' s integral representation of the third kind

$$f(z_1, z_2) = f(0,0) + \sum_{\nu=1}^{2} \frac{z_\nu}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} \frac{f'_\nu(r_1(\tau)\zeta,\ r_2(\tau)\zeta e^{-it})\, d\zeta}{\zeta - u} \quad (1)$$

([2], Theorem 3.1)*. Thus the problem is solved of establishing the general Temlyakov integral representation in the case $n = 2$ [2] (in [1] this is formula (3) for $n = 2$), consisting of Temlyakov integral representations of the first, second, and third kinds. The formula (3) obtained by the author in [1] (in the author' s paper [2] this is formula (7.1)) solves the problem of the complete extension of Temlyakov' s general integral representation to the case $n > 2$. Similarly, the formulas (8.1) and (9.1) established by the author in [2] are solutions of the problems of establishing, in the case $n = 2$, the general integral representations of Poisson–Temlyakov and Schwarz–Temlyakov, respectively, and of their complete extension to the case $n > 2$.

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* By an operator of zero order applied to a function we mean the function itself.

** Incidentally, let us note that, as follows from (1) (see also (2), §7), the most important feature of Temlyakov' s integral formula is that the last inner integral in this formula is either a Cauchy integral ($k = n-1$) in the case of one complex variable, or the corresponding operator of this integral.

3. The introduction by the author [2-5] of inverse operators into Temlyakov integral representations** made it possible for the author (see [2], § 6; [3-5]) to obtain general integral representations which, while preserving a close connection with the Cauchy integral of one complex variable, are still more subordinated to the specific character of complete n-circular domains; another peculiarity of these representations, as noted in [14], is that the behavior of the integrals of the type formed on the basis of the integral representations entering into these general integral representations, generally speaking, has qualitative differences from the behavior of integrals of Temlyakov' s type of the first kind [11, 15]. At the same time, as is not difficult to see, the indicated general integral representations serve as a source for the formation of new quasianalytic functions in Temlyakov' s sense [16].
4. The operator method developed by the author in the theory of integral representations in the case of bounded convex complete n-circular domains ($n \geq 2$) can, within certain limits, also be applicable in the theory of integral representations in the case of considerably broader classes of domains of the space C^n, $n \geq 2$: see, for example, Theorem 3 of [1] and Theorem 5 of [3]. Theorems 3 [1] and 5 [3] of the author establish, in the case of convex domains of the space C^n ($n \geq 2$), respectively, the formulas (in [1, 3] these formulas were not written out for brevity of exposition)

$$\underset{z\in D}{F(z)} = \alpha F(0) + \frac{1}{n+\alpha(1-n)} \sum_{\nu=1}^{n} \frac{z_\nu^\alpha}{(2\pi i)^n} \int_{\partial D} \lambda^{1-n} \times$$

$$\times L\begin{pmatrix} n-k-1-\alpha \\ m_{\alpha+1}\ m_{n-k-1} \end{pmatrix} \left[\left(\sum_{l=1}^{n} (\xi_l - z_l)\Phi'_{\xi_l}\right)^{-1}\right]^{(k)} L\begin{pmatrix} m_{n-k} \\ m_{n-1} \end{pmatrix} [F_\nu^{(\alpha)}(\xi)] \sum_{l=1}^{n} \sigma[l]\, d\bar{\xi}[l] \wedge d\xi, \tag{2}$$

$$\underset{z\in D}{F(z)} = \alpha F(0) + \frac{1}{n+\alpha(1-n)} \sum_{\nu=1}^{n} \frac{z_\nu^\alpha}{(2\pi i)^n} \int_{\partial D} \lambda^{1-n} \times \tag{3}$$

$$\times L\begin{pmatrix} n-1-\alpha \\ \alpha+1,\ n-1 \end{pmatrix} \left[L\begin{pmatrix} -k \\ \gamma_1 \end{pmatrix} \left[\left(\sum_{l=1}^{n} (\xi_l - z_l)\Phi'_{\xi_l}\right)^{-1}\right]\right] L\begin{pmatrix} l_\nu \\ \gamma_1 \end{pmatrix} [F_\nu^{(\alpha)}(\xi)] \sum_{l=1}^{n} \sigma[l]\, d\bar{\xi}[l] \wedge d\xi.$$

$$\text{Here} \quad u = \frac{\tau}{r_1(\tau)} z_1 + \frac{1-\tau}{r_2(\tau)} z_2 e^{it}.$$

Formula (1) has an analogous form also in the case of a domain $D = (T)$, $n = 2$ (see [1], [2], Corollary 7.4).

** Concerning inverse operators in [2, 3], see [4, 5].

Here $z = (z_1, \dots, z_n)$, $\zeta = (\zeta_1, \dots, \zeta_n)$,

$$\lambda = \sum_{l=1}^{n} \zeta_l \Phi'_{\zeta_l},$$

$\sigma[l]$ is the determinant in whose first row stand $\Phi'_{\zeta_1}, \dots, \Phi'_{\zeta_n}$, and in the remaining rows the derivatives of the indicated functions with respect to $\overline{\zeta}_1, \dots, \overline{\zeta}_{l-1}, \overline{\zeta}_{l+1}, \dots, \overline{\zeta}_n$, respectively,

$$d\zeta = d\zeta_1 \wedge \dots \wedge d\zeta_n, \qquad d\overline{\zeta}[l] = d\overline{\zeta}_1 \wedge \dots \wedge d\overline{\zeta}_{l-1} \wedge d\overline{\zeta}_{l+1} \wedge \dots \wedge d\overline{\zeta}_n,$$

where $\wedge$ is the symbol of exterior multiplication (for all other explanations see (1,3)). Formula (2) is a solution of the problem of establishing, for $n = 2$, a general integral representation in the case of convex domains corresponding to Temlyakov' s general integral representation, and of its complete extension to the case $n > 2$. In the case $n = 2$, from formula (2), for $\alpha = 0$ ($k = 1$) and $\alpha = 1$, we obtain integral representations corresponding to Temlyakov' s integral representations of the first and third kinds, respectively. We note that, analogously to Corollary 5.2 in (5), it is established that formula (2) is a consequence of formula (3).

5. In the case of the disk, the author (2,3,5) also obtained general integral representations closely connected with the Cauchy integral.* Their distinctive feature, as is not difficult to see, is that the behavior of type integrals formed on the basis of the integral representations entering into these general integral representations, for which $\alpha + k > 0$,** generally speaking, has qualitative differences (nonanalyticity outside the disk, but quasianalyticity there in the sense indicated above) from the behavior of the Cauchy type integral. Consequently, the behavior of these type integrals is similar to the behavior of Temlyakov type integrals and to the behavior of the type integrals considered in (14). It is therefore natural to regard, in the above-mentioned case of the disk, the integral representations (except for the case $\alpha + k = 0$) as an analogue, in the case $n = 1$, of Temlyakov' s integral representations and of the integral representations discussed in item 3. Thus, in the case of the disk, these integral representations (except for the case $\alpha + k = 0$) solve the problem of establishing, in the case of one complex variable, analogues of Temlyakov' s integral representations and of the integral representations noted in item 3.

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* They remain valid also in the case of star-shaped domains of the space C^1 (see ([2]), Remark 5.2; ([3]); ([5]), Remark 2.1).

** In ([2,3]), in the integral representations in the case of one variable, α is taken equal to zero.

Note: Figure translations are in progress. See original paper for figures.

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