

# GENERALIZED EQUIPARAMETRIC MANIFOLDS IN A MULTIDIMENSIONAL PROJECTIVE SPACE

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## GENERALIZED EQUIPARAMETRIC MANIFOLDS IN A MULTIDIMENSIONAL PROJECTIVE SPACE

*(Presented by Academician A. D. Aleksandrov on 10 VII 1967)*

As is known <sup>(1)</sup>, the problem of invariantly equipping an  $m$ -surface  $S_m$  in an  $n$ -dimensional projective space  $P_n$  is the construction of a field of  $(n - m)$ -planes invariantly associated with  $S_m$ . Therefore it is of interest to study the  $m$ -parametric manifold  $E(0, n - m, m)$ , whose element is an  $(n - m)$ -plane  $L_{n-m}$  (the basic  $(n - m)$ -plane) with a fixed point  $L$  in it. In a deeper study of the geometry of  $S_m$  in  $P_n$ , an essential role is played by the manifold  $E(L, \hat{L}_m, \hat{L}_{m+1})$ , whose element consists of  $n - m$  linearly independent  $(m + 1)$ -planes  $L_{\hat{m}+1}$  ( $\hat{\alpha} = m + 1, \dots, n$ ), passing through the  $m$ -plane  $L_m$  in which the point  $L$  is given, where  $L_m$  is the tangent  $m$ -plane of the  $m$ -surface  $S_m$  described by the point  $L$ . These manifolds are the subject considered in the present work. Analogous equiparametric manifolds in three-dimensional space were studied in <sup>(2)</sup>.

1. The derivation formulas of a certain frame, consisting of analytic points  $A_0, A_1, \dots, A_n$ , in the  $n$ -dimensional projective space  $P_n$  will be written in the form  $dA_i = \omega_i^k A_k$  ( $i, k = 0, 1, \dots, n$ ), where  $\omega_i^k$  are Pfaffian differential forms satisfying the structure equations  $D\omega_i^k = [\omega_i^j \omega_j^k]$  ( $j = 0, 1, \dots, n$ ) and the relation  $\omega_i^i = 0$ . An analytic fixation of a semicanonical frame in the sense of <sup>(3, 4)</sup> of the manifold  $E(0, n - m, m)$  in  $P_n$  gives  $\omega_i^k = \Lambda_{i\alpha}^k \omega_\alpha^0$  ( $\alpha = 1, 2, \dots, m$ ), where  $\Lambda_{i\alpha}^k$  satisfy the relations

$$\Lambda_{0\beta}^\alpha = \delta_{\beta}^\alpha, \quad \Lambda_{0\beta}^{\hat{\alpha}} = 0, \quad \Lambda_{\alpha\beta}^{\hat{\beta}} = \Lambda_{\beta\alpha}^{\hat{\beta}}, \quad A_{\underbrace{0 \dots 0}_{m-1} \hat{\alpha}} = 0, \quad A_{\underbrace{\hat{\alpha} \dots \hat{\alpha}}_{m-1} \hat{\beta}} = 0 \quad (\hat{\alpha} \neq \hat{\beta}),$$

$$A_{\underbrace{\hat{\alpha} \dots \hat{\alpha}}_m} = m, \quad \Lambda_{n\beta}^0 = 0, \quad \Lambda_{m+1, \alpha}^0 = 1$$

$$(\alpha, \beta, \gamma = 1, \dots, m; \quad \hat{\alpha}, \hat{\beta} = m + 1, \dots, n),$$

$$\Lambda_{n\beta}^\gamma \Lambda_{\gamma\alpha}^0 - \Lambda_{n\alpha}^\gamma \Lambda_{\gamma\beta}^0 = \Lambda_{\hat{\beta}\beta}^0 \Lambda_{n\alpha}^{\hat{\beta}} - \Lambda_{\hat{\beta}\alpha}^0 \Lambda_{n\beta}^{\hat{\beta}} \quad (\beta \neq \alpha, \hat{\beta} \neq n).$$

Here it is assumed that  $m > 2$  and  $n < m(m + 1)$ , and the quantities

$$m! A_{\tilde{\alpha}_1 \tilde{\alpha}_2 \dots \tilde{\alpha}_m} = \Lambda_{(\tilde{\alpha}_1|1}^1 \Lambda_{\tilde{\alpha}_2|2}^2 \dots \Lambda_{\tilde{\alpha}_m|m}^m) \quad (\tilde{\alpha}_1, \dots, \tilde{\alpha}_m = 0, m + 1, \dots, n)$$

are absolutely symmetric in all indices. The manifold  $E(0, n - m, m)$ , referred to an arbitrary system of one-dimensional submanifolds, is determined with an arbitrariness of  $m^2 + n(m - 1) - 3n$  functions of  $m$  arguments. The constructed semicanonical frame has the following geometric characteristic. The point  $A_0$  coincides with the point  $L$ , and  $L_{n-m} = (A_0 A_{m+1} \dots A_n)$  is the basic  $(n - m)$ -plane. The  $(n - m - 1)$ -plane  $L_{n-m-1} = (A_{m+1} \dots A_n)$  is the  $(m - 1)$ -st polar, in the sense of <sup>(5, 6)</sup>, of the point  $A_0$  with respect to the focal algebraic hypersurface  $\Phi_{n-m-1}^m$  of order  $m$  of the basic  $(n - m)$ -plane. The algebraic surface  $\Psi_{n-m-2}^m$  of order  $m$  and dimension  $n - m - 2$ , belonging simultaneously to  $\Phi_{n-m-1}^m$  and

$L_{n-m-1}$  is called a polar algebraic surface. The points  $A_{\hat{\alpha}}$  ( $\hat{\alpha} = m + 1, \dots, n$ ) in  $L_{n-m-1}$  are chosen so that  $A^{\hat{\alpha}} = (A_{m+1} \dots A_{\hat{\alpha}-1} A_{\hat{\alpha}+1} \dots A_n)$  is the  $(m - 1)$ -st polar of each of these points  $A_{\hat{\alpha}}$  with respect to the polar algebraic surface. The  $(m - 1)$ -plane  $L_{m-1} = (A_1 A_2 \dots A_m)$  belongs simultaneously to the tangent to  $S_m$  at the point  $A_0$ , to the  $m$ -plane  $L_m = (A_0 - A_m)$ , and to the hyperplane  $Q_{n-1}$  passing through  $L_{n-m-1}$  and through the tangent  $m$ -plane to the  $m$ -surface described by the point  $A_n$ . The lines  $A_0 A_{\alpha}$  are tangent to the lines described by the point  $A_0$  under displacements along the one-dimensional coordinate submanifolds. The points  $A_{\alpha}$  are the points of intersection of the lines  $A_0 A_{\alpha}$  with the  $(m - 1)$ -plane  $L_{m-1}$ .

Let us note two special classes of systems of submanifolds of the manifold  $E(0, n - m, m)$ .

**A.** A system  $S$  of submanifolds, whose assignment is characterized by the relations  $\Lambda_{\hat{\alpha}\beta}^{\alpha} = 0$ ,  $\Lambda_{\hat{\alpha}\alpha}^{\beta} \neq \Lambda_{\hat{\alpha}\beta}^{\alpha}$  ( $\alpha \neq \beta$ , no summation over  $\alpha$  and  $\beta$ ,  $\hat{\alpha}$  fixed), consists of one-dimensional submanifolds corresponding to the focal displacements of different foci of the algebraic surface  $\Phi_{n-m-1}^m$  belonging to the line  $A_0 A_{\hat{\alpha}}$ . The canonical frame obtained by referring the manifold  $E(0, n - m, m)$  to the system  $S$  is called the principal frame.

**B.** A system  $\Gamma$  of submanifolds, whose assignment is characterized by the relations  $\Lambda_{\alpha\beta}^{\hat{\alpha}} = \Lambda_{\alpha\beta}^{\hat{\beta}} = 0$ ,  $\Lambda_{\alpha\alpha}^{\hat{\alpha}} \Lambda_{\beta\beta}^{\hat{\beta}} \neq \Lambda_{\alpha\alpha}^{\hat{\beta}} \Lambda_{\beta\beta}^{\hat{\alpha}}$  ( $\alpha \neq \beta$ ,  $\hat{\alpha} \neq \hat{\beta}$ ,  $\alpha$  and  $\hat{\beta}$  fixed), consists of one-dimensional submanifolds corresponding to the focal displacements of different hyperplanes  $L_{n-1}^{\hat{\alpha}}$ ,  $\hat{\alpha}, \hat{\beta} = (L_m A_{m+1} \dots A_{\hat{\alpha}-1} A_{\hat{\alpha}+1} \dots A_{\hat{\beta}-1} A_{\hat{\beta}+1} \dots A_n, \Lambda_{\alpha\alpha}^{\hat{\alpha}} A_{\hat{\alpha}} + \Lambda_{\alpha\alpha}^{\hat{\beta}} A_{\hat{\beta}})$  (no summation over  $\hat{\alpha}$  and  $\hat{\beta}$ ), containing under these displacements the first differential neighborhood of the  $m$ -plane  $L_m$  and passing through the  $(m + 1)$ -planes  $L_{m+1}^{\hat{\gamma}} = (L_m A_{\hat{\gamma}})$  ( $\hat{\gamma} = m + 1, \dots, n$ ;  $\hat{\gamma} \neq \hat{\alpha}, \hat{\beta}$ ). The canonical frame obtained by referring the manifold  $E(0, n - m, m)$  to the system  $\Gamma$  is called a  $\Gamma$ -frame.

The systems  $S$  and  $\Gamma$  make it possible to distinguish the following special classes of manifolds  $E(0, n - m, m)$ .

**I.** A manifold  $E(0, n - m, m)$  having, in the principal or  $\Gamma$ -frame, natural equations  $\Lambda_{\alpha\alpha}^{\beta} = 0$  ( $\alpha \neq \beta$ ), is defined with arbitrariness  $(m + 1)(n - m)$  functions of  $m$  arguments and is characterized by the fact that the osculating 2-planes  $a_{\alpha}$  to the coordinate lines of the surface  $S_m$  described by the point  $A_0$  intersect the principal  $(n - m)$ -plane  $L_{n-m}$  along straight lines  $l_{\alpha} = \Lambda_{\alpha\alpha}^{\hat{\alpha}}(A_0 A_{\hat{\alpha}})$ , i.e., in accordance with (7), the coordinate lines of the surface  $S_m$  are geodesic lines of the projective-connection space induced by the  $(n - m)$ -plane  $L_{n-m}$  along  $S_m$ .

**II.** A manifold  $E(0, n - m, m)$  having, in the principal frame, natural equations  $\Lambda_{\alpha\alpha}^{\hat{\beta}} = 0$ , is defined with arbitrariness  $m^2 - 2m + n$  functions of  $m$  arguments and is characterized by the fact that the coordinate lines of the surface  $S_m$  are asymptotic lines.

**III.** A manifold  $E(0, n - m, m)$  having, in the principal frame, natural equations  $\Lambda_{\alpha\alpha}^{\beta} = 0$ ,  $\Lambda_{\gamma\gamma}^{\hat{\alpha}} = 0$  ( $\alpha \neq \beta$ ,  $\alpha, \beta, \gamma = 1, \dots, m$ ), is defined with arbitrariness  $n - m$  functions of  $m$  arguments and is characterized by the fact that the coordinate lines of the surface  $S_m$  are straight lines.

**2. Fixing** the semicanonical frame of the manifold  $E(L, L_m, L_{m+1}^{\hat{\alpha}})$ , carried out analytically, gives  $\omega_i^k = \lambda_{i\alpha}^k \omega^{\alpha}$ , where  $\Lambda_{i\alpha}^k$  satisfy the relations

$$\Lambda_{0\beta}^{\alpha} = \delta_{\beta}^{\alpha}, \quad \Lambda_{0\beta}^{\alpha} = 0, \quad \Lambda_{\alpha\beta}^{\hat{\beta}} = \Lambda_{\beta\alpha}^{\hat{\beta}}, \quad \Lambda_{\hat{\gamma}\alpha}^n = 0, \quad \Lambda_{n\alpha}^{m+1} = 0, \quad A^{\hat{\alpha}\dots\hat{\alpha}} = m (\hat{\gamma} \neq n),$$

$$\Lambda_{m+1,\beta}^{\alpha} \Lambda_{\alpha\gamma}^{m+1} + \Lambda_{\gamma\gamma}^{m+1} \Lambda_{n\beta}^{\hat{\gamma}} = \Lambda_{m+1,\gamma}^{\alpha} \Lambda_{\alpha\beta}^{m+1} + \Lambda_{\gamma\beta}^{m+1} \Lambda_{n\gamma}^{\hat{\gamma}} \quad (\hat{\gamma} \neq n, \gamma \neq \beta),$$

$$\Lambda_{\alpha\gamma}^n \Lambda_{\alpha\beta}^{\alpha} = \Lambda_{\alpha\gamma}^n \Lambda_{\alpha\gamma}^{\alpha}, \quad \Lambda_{\beta\alpha}^{\alpha} = 0, \quad \Lambda_{n\beta}^0 = 0, \quad \Lambda_{m+1,\alpha}^0 = 1 \quad (\beta \neq \gamma, \hat{\alpha} \neq n),$$

$$\Lambda_{n\beta}^{\gamma} \Lambda_{\gamma\tau}^0 - \Lambda_{n\tau}^{\gamma} \Lambda_{\gamma\beta}^0 = \Lambda_{\hat{\beta}\beta}^0 \Lambda_{n\tau}^{\beta} - \Lambda_{\beta\tau}^0 \Lambda_{n\beta}^{\hat{\beta}} \quad (\hat{\beta} \neq m + 1, \beta \neq \tau).$$

Here  $m > 2$ , and the quantities

$$m! A^{\hat{\alpha}_1 \dots \hat{\alpha}_m} = \Lambda_{1[1}^{\hat{\alpha}_1} \Lambda_{2]2}^{\hat{\alpha}_2} \dots \Lambda_{[m]m}^{\hat{\alpha}_m} \quad (\hat{\alpha}_{\beta} = m + 1, \dots, n)$$

are absolutely symmetric in all indices. The manifold  $E(0, n - m, m)$ , referred to an arbitrary system of one-dimensional submanifolds, is determined with arbitrariness  $(n - m)^2 + m(m - 1)$  functions of  $m$  arguments. In this frame the point  $A_0$  coincides with the point  $L$ ,  $L_m = (A_0 A_1 \dots A_m)$ , and  $L_{m+1}^{\hat{\alpha}} = (L_m A_{\hat{\alpha}})$

( $\hat{\alpha} = m + 1, \dots, n$ ). The lines  $A_0 A_{\hat{\alpha}}$  ( $\hat{\alpha} \neq n$ ) are  $B$ -characteristics of the  $(m + 1)$ -planes  $L_{m+1}^{\hat{\alpha}}$  ( $\hat{\alpha} \neq n$ ) with respect to  $L_{n-1}^n = (L_{mA_{m+1}} \dots A_{n-1})$ , i.e., the entire first differential neighborhood of each of these lines, belonging to  $L_{m+1}^{\hat{\alpha}}$  ( $\hat{\alpha} \neq n$ ), does not leave  $L_{n-1}^n$ . The line  $A_0 A_n$  is a  $B$ -characteristic of the  $(m + 1)$ -plane  $L_{m+1}^n$  with respect to  $L_{n-1}^{m+1} = (L_{mA_{m+2}} \dots A_n)$ . Therefore the  $(n - m)$ -plane  $L_{n-m} = (A_0 A_{m+1} \dots A_n)$  passes through the lines  $A_0 A_{\hat{\alpha}}$ . Consequently, with each element of the manifold  $E(L, L_m, L_{m+1}^{\hat{\alpha}})$  there is invariantly associated the manifold  $E(0, n - m, m)$ , and the further geometric characteristic of the frame is evident. The manifold  $ES_m$  is the manifold  $E(L, L_m, L_{m+1}^{\hat{\alpha}})$  determined by the natural equations

$$A^{\hat{\alpha} \dots \hat{\alpha}} = 0 \quad (\hat{\alpha} \neq \hat{\beta}).$$

This manifold is determined with arbitrariness the product of  $n - m$  functions of  $m$  arguments, where

$$m + 2 < n < \frac{1}{2}m(m + 3).$$

The totality of all focal hyperplanes <sup>(8,9)</sup> of the  $m$ -surface  $S_m$  forms a certain hypercone  $T_{n-1}^m$  of class  $m$  with vertex  $L_m$ , defined by the equation

$$T \equiv A^{\hat{\alpha}_1 \hat{\alpha}_2 \dots \hat{\alpha}_m} x_{\hat{\alpha}_1} x_{\hat{\alpha}_2} \dots x_{\hat{\alpha}_m} = 0.$$

For the manifold  $ES_m$  we have: each  $(m + 1)$ -plane  $L_{m+1}^{\hat{\alpha}} = (L_{mA_{\hat{\alpha}}})$  is an  $(m - 1)$ -fold plane in the sense of <sup>(5,6)</sup> of the hyperplane

$$A^{\hat{\alpha}} = (L_{mA_{m+1}} \dots A_{\hat{\alpha}-1} A_{\hat{\alpha}+1} \dots A_n)$$

with respect to  $T_{n-1}^m$ . Let us note that in the manifold  $ES_m$  the surface  $S_m$  is arbitrary. Therefore the semicanonical frame of the manifold  $ES_m$  may be regarded as the semicanonical frame of an arbitrary  $m$ -surface  $S_m$  in  $P_n$  ( $m + 2 < n < \frac{1}{2}m(m + 3)$ ).

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## CITED LITERATURE

- <sup>1</sup> G. F. Laptev, *Itogi nauki, Geometry*, 1963, p. 5.
- <sup>2</sup> Gh. Gheorghiev, I. Popa, *Anal. stiint. ale Univ. Iasi, Sect. I*, VII, 2, 283 (1961).

- <sup>3</sup> R. N. Shcherbakov, *Geom. sborn.*, 3 (Trans. Tomsk Univ., 168), 5 (1963).
- <sup>4</sup> K. Svoboda, V. Havel, J. Klapka, *Comm. Math. Univ. Carolinae*, 5, 4, 183 (1964).
- <sup>5</sup> G. Castelnuovo, F. Enriques, *Encyklopädie der math. Wiss.*, III, 2, 1907-1910, pp. 636-637.
- <sup>6</sup> G. B. Gurevich, *Foundations of the Theory of Algebraic Invariants*, Moscow-Leningrad, 1948.
- <sup>7</sup> M. A. Akivis, *Matem. sborn.*, 58 (100), 2, 695 (1962).
- <sup>8</sup> M. A. Akivis, *Izv. vyssh. uchebn. zaved., Matematika*, No. 1, 9 (1957).
- <sup>9</sup> M. A. Akivis, *DAN*, 146, No. 3, 515 (1962).

*Note: Figure translations are in progress. See original paper for figures.*

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